This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

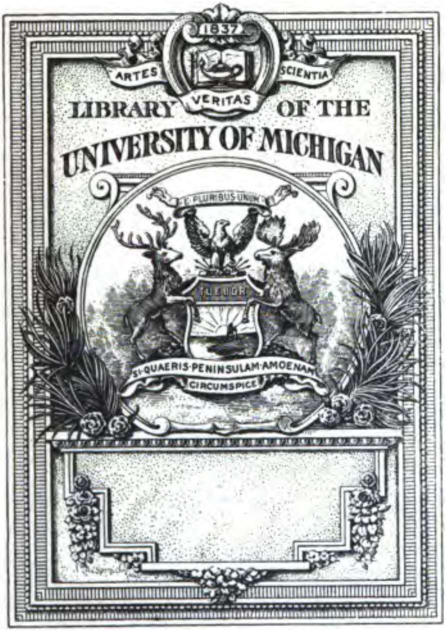
Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at http://books.google.com/


1837
ARTES
SCIENTIA
VERITAS
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QC
127
.M66

4/6

BELL'S SCIENCE SERIES

Edited by

PERCY GROOM, M.A.(Cantab. et Oxon.), F.L.S.
Lecturer on Botany in the Royal Indian Engineering College, Cooper's Hill

AND

G. M. MINCHIN, M.A., F.R.S.
Professor of Applied Mathematics in the same College

THIS Series has been designed to supply the wants of Students in the upper forms of Schools, and Candidates for the various examinations, not distinctly elementary, at which scientific subjects are now set.

ELEMENTARY BOTANY. By PERCY GROOM, M.A., F.L.S., sometime Examiner in Botany to the University of Oxford. Second Edition. With 275 Illustrations. Crown 8vo, 3s. 6d.

AN INTRODUCTION TO THE STUDY OF THE COMPARATIVE ANATOMY OF ANIMALS. By G. C. BOURNE, M.A., Fellow and Tutor of New College, Oxford. With numerous Illustrations. Vol. I. Animal Organisation. The Protozoa and Cœlenterata. 4s. 6d.

THE STUDENT'S DYNAMICS. Comprising Statics and Kinetics. By Professor G. M. MINCHIN, M.A., F.R.S. 3s. 6d.

ELEMENTARY GENERAL SCIENCE. By D. E. JONES, D.Sc., Science Inspector, formerly Professor of Physics in the University College of Wales, Aberystwith, and D. S. M'NAIR, Ph.D., B.Sc. [*In the Press.*

PHYSIOGRAPHY. By H. N. DICKSON, F.R.S.E., F.R.Met. Soc., F.R.G.S. [*In the Press.*

CHEMISTRY. By JAMES WALKER, D.Sc., Professor of Chemistry in University College, Dundee.

ELECTRICITY AND MAGNETISM. By OLIVER J. LODGE, D.Sc., F.R.S., LL.D., M.I.E.E., Professor of Physics in University College, Liverpool.

LONDON: GEORGE BELL & SONS

THE STUDENT'S DYNAMICS

GEORGE BELL & SONS

LONDON: YORK STREET, COVENT GARDEN

NEW YORK: 66 FIFTH AVENUE, AND

BOMBAY: . . . . 53 ESPLANADE ROAD

CAMBRIDGE: DEIGHTON BELL AND CO.

THE STUDENT'S DYNAMICS

COMPRISING

STATICS AND KINETICS

BY

eorge inchin

G. M. MINCHIN, M.A., F.R.S.

PROFESSOR OF APPLIED MATHEMATICS IN
THE ROYAL ENGINEERING COLLEGE
COOPER'S HILL

LONDON
GEORGE BELL & SONS
1900

PREFACE

THIS elementary treatise will be found to possess two main characteristics. In the first place, it treats the Science of Force, or Dynamics, as founded directly on Newton's Axioms, or Laws, of Motion—and more particularly on the Second Axiom. That is to say, the notion of *force* which is adopted at the outset is that which measures it by the *acceleration* which it can produce in a particle of given mass. Hence no distinction whatever is made between *force as it is dealt with in Statics* and *force as it is dealt with in Kinetics*—"statical force" and "dynamical force," as the works and teachers of a past generation most unscientifically and misleadingly expressed it.

Statics and Kinetics here go together from the very outset, and constitute the two divisions of Dynamics. The student is taught that they deal with the very same entity (force), measured in the same way in both.

This procedure is, of course, no novelty; for other treatises (few in number, however) had previously adopted it; and I am convinced that for the future it should be the rule instead of the exception.

The second main characteristic of the work is the great prominence given from the beginning, and throughout, to *arithmetical illustration and calculation.*

Long experience in teaching has convinced me of the great mistake made by teachers—especially in the old universities—in expounding the principles of a physical subject *first* by algebraical symbols, followed (if at all) by a few numerical calculations. The very reverse of this method is alone efficacious.

Arithmetic is the reality of every science; and the principles of every science are far more readily and tenaciously grasped when they are applied to definite, concrete, arithmetical ex-

amples than when they are presented to the learner in the shape of algebraical symbols.

I hold that unless a mathematical physicist can bring his knowledge of Electricity, Magnetism, Hydrodynamics, or any other branch of Science, down to arithmetical calculation, his knowledge is unsound and useless.

It will, then, be found throughout this work that the principles of every new part of the subject are always introduced by numerical, and not by algebraical, illustration: I make Algebra follow, not precede, Arithmetic.

So far as the amount of mathematical knowledge assumed on the part of the learner is concerned, I may say that I require the ordinary school-boy knowledge of Geometry and Algebra, together with the very rudiments of Trigonometry. Indeed, the knowledge of the properties of the sine, cosine, tangent, etc., of one angle is all that I require in Trigonometry.

In inviting students to commence the study of Dynamics without this slender trigonometrical foundation, the University of London has, in my opinion, made a most unfortunate mistake.

What can a student really know about resolving or compounding forces, or taking moments, if he does not know what a *cosine* means?

Examiners laugh at this attempt to proceed without the elements of Trigonometry, and are sometimes at their wit's end to avoid setting questions which violate this inconvenient restriction.

If the system is radically unsound—as I have no doubt that it is—it will be readily seen that large examining bodies, in adopting it, are exercising an evil influence on the school teaching of the country.

Proceeding to matters of more detail, it will be observed that I have dwelt very much on the subjects of Impulse and Momentum and of Work and Energy. The student who masters thoroughly the Dynamics of Pile Driving, or of the firing of a shot from a gun into a block of wood, may be said to understand nearly all that is important in the Dynamics of a particle.

Finally, in this work I have made another effort to banish that extremely misleading term *centrifugal force*. It is an evil

as great in this part of Physics as the terms *electromotive force* and *magnetomotive force* (which are not forces at all) are in Electricity and Magnetism.

The proofs were read over, and in several instances corrected, by my colleague, Professor Alfred Lodge, to whom I beg to tender thanks.

GEORGE M. MINCHIN.

ERRATA

In Example 5, page 56, *for* (*a*) 39 pounds' weight; (*b*) 26 pounds' weight, *read* (*a*) $\frac{89}{13}$ pounds' weight; (*b*) $\frac{34}{13}$ pounds' weight.

In Example 7, page 72, *for* $\alpha = g$ *read* $3\alpha = g$.

In Example 11, page 81, *for* distance from $O = 150$ feet *read* distance from $O = 250$ feet.

In Example 5, page 117, *for* $81\frac{118}{178}$ *read* $83\frac{79}{118}$.

CONTENTS

ADDITIONAL ERRATA

In p. 80, *line* 8 *from end, for* "starting *B*" *read* "starting-point."
„ 124, „ 10 „ *top, insert* "is" *before* "involved."
„ 127, „ 5 „ *end, for* "69" *read* "68."

ELEMENTARY DYNAMICS

CHAPTER I

COMPOSITION AND RESOLUTION OF VELOCITIES

1. **Units of Length, Time, and Mass.**—Calculations in the physical sciences—of which Dynamics is, of course, one—require us to adopt units in measuring the three fundamental things with which these sciences deal—namely, *space*, *time*, and *mass*.

As regards *Space*, everything can be measured if we take a unit length. Thus, if we take a *foot* as our unit of length, we can measure any *length* as a number of feet, any *area* as a number of square feet, and any *volume* as a number of cubic feet. Sometimes a foot is a convenient unit of length; sometimes a *yard* (*i.e.* 3 feet) is more convenient.

The discussion of units is a rather tedious subject, so that we shall be very brief about it here. Suffice it to say that there is another unit of length, about the same as a yard, which is in very common use in Continental countries—namely, the *mètre*. If a great semi-circle is drawn on the Earth's surface from the North Pole to the South Pole, and half of this meridian is taken, we have what is called an *earth-quadrant*. Now, if this earth-quadrant is divided into 10 millions of equal parts, each part is called a *mètre*. This mètre is about

3 feet 3 inches and $\frac{3}{8}$ of an inch,

or a little more than the English yard.

If a mètre is divided into 10 equal parts, each part is called a *decimètre*; if a decimètre is divided into 10 equal parts, each of these parts is $\frac{1}{100}$ of a mètre, and is called a *centimètre*; if a centimètre is divided into 10 equal parts, each is called

a *millimètre*, which, of course, is $\frac{1}{1000}$ of a mètre. It is useful to remember that—

1 inch = 2·54 centimètres, nearly.

As a unit of *Time* may be taken 1 second, or 1 minute, or 1 hour, or 1 year, or 1 century, according to convenience. The time taken by the Earth to rotate once round its axis—*i.e.* a day, is divided into 24 hours; each hour is divided into 60 minutes, and each minute into 60 seconds. There is, of course, no reason why the day might not be divided into 50, 100, or any other number of "hours"; but the division into 24, and then into 60 parts is adopted everywhere.

Finally, we come to *Mass*. A common unit of mass in England is the *pound*, which cannot be otherwise defined than by saying that a specimen of it, kept as a standard, is to be seen in the Exchequer Office.

Another, and much more easily defined, unit of mass is the *gramme*. Imagine a cube to be made having each of its edges 1 centimètre long. Now water, when it is very nearly, but not quite, ice-cold—or, more exactly, when its temperature is about 4 degrees on the centigrade scale—has its greatest density; that is, its particles are most closely packed together. Imagine the little cubic centimètre to be exactly filled with water in this state; then the quantity of matter which is contained in the cubic centimètre is called a *gramme mass*.

A portion of any other kind of matter—wood, lead, gold, etc.—which has the same weight as that of the cubic centimètre of water contains also 1 gramme mass.

The gramme is such an important unit that we formally give its definition:—*A gramme mass is the quantity of matter contained in* 1 *cubic centimetre of pure water in its most dense state.*

For many purposes the gramme is much too small a unit; so that 10, or 100, or 1000 times its mass is often taken. A mass of 1000 grammes—called a *kilogramme*—is a very common unit of mass. A kilogramme is about $2\frac{1}{4}$ pounds.

It is important to remember the fact *that ice, though solid, is not the most dense state of water*; so that, although the

surface of a lake may be a mass of ice, the fishes underneath may be in water considerably warmer than ice; for, since water whose temperature is 4° C, or about 39° Fahrenheit, is more dense than ice (whose temperature is 0° C, or 32° F), the ice would float on the top of this comparatively warm water.

2. **Velocity.**—Suppose that a point moves along a right line Ox (fig. 1), in such a way that it always takes *the same time to move over the same length*; then we have what is called *uniform motion.* For example, suppose that the moving point always takes 5 seconds to move over 30 feet of the line Ox, no matter where the 30 feet are taken; then its rate of motion may be described as a rate of 30 feet per 5 seconds, or 6 feet per second. This rate of moving is called the *velocity* of the point. Observe that, before we can say that the point's rate of moving is always the same, we must know that it describes the length of 30 feet in 5 seconds no matter where the extremities of the 30 feet are taken on the line Ox. Thus, if $OP=PQ=QR=30$ feet, the point takes 5 seconds to move over OP, and the same time to move over PQ, and the same time to move over QR. But the lengths of 30 feet are not to be taken merely end-on to each other, as OP, PQ, QR are; for we may take any such point as A and another, B, such that $AB=30$ feet, and it must still be true that the time of going over AB is 5 seconds, or the motion would not be uniform.

Fig. 1.

Remember, then, that—

Velocity means *time-rate of describing length.*

Thus, when a railway train is moving uniformly in such a way that it always takes 1 minute to move over half a mile, we may speak of its velocity as one of half a mile per minute, or 880 yards per minute, or 44 feet per second, or 30 miles per hour. Velocity is "length per time," and it does not matter in what units we measure length (whether feet, yards, mètres, or miles), or in what units we measure time (seconds, minutes, or hours).

If the moving point does not move over the same distance

in the same time throughout its path, its velocity is not uniform. To each position of the point belongs a special velocity the magnitude of which can be found with a high degree of accuracy by the simple rule: take the distance gone over by the point in the $\frac{1}{1000}$ of a second and multiply this distance by 1000. Thus if from the position P the point moves to the position Q, a distance of $\frac{1}{10}$ inch, in $\frac{1}{1000}$ second, the velocity at P is very nearly 100 inches per second. A still better estimate of the rate of moving at P would be got by taking the distance, PQ, moved over in the one-millionth of a second and multiplying PQ by a million (or dividing PQ by the time, $\frac{1}{1000000}$ second).

In this way, no matter how variable the rate of moving of a point may be, we obtain a clear notion of the way in which its value in any position of the point may be found.*

3. **Acceleration.**—If a moving point does not, in different parts of its path, always move over the same length in the same time, its motion, of course, is not uniform. Now, let us suppose a very simple case of motion of a point whose motion is not uniform. Imagine that we can by some means measure the velocity of the point at any instant we please, and let us make measurements at the end of (say) every 5 seconds. Suppose that we find velocities of—

$$3, 18, 33, 48, 63, 78, \dots$$

feet per second at the end of—

$$5, 10, 15, 20, 25, 30, \dots$$

seconds.

Here, then, the velocity of the point is variable, but it varies in a very simple manner. What is the law according to which it varies? We see that each of the above velocities exceeds the one before it by 15 feet per second; that is to

* An amusingly fallacious definition of the velocity, in any position, of a point whose motion is variable is given in almost all text-books. It is this:—"The velocity of a point, when variable, is measured by the distance which would be passed over by the point in a unit of time, if it continued to move during that unit of time with *the velocity it has at the instant under consideration*"—the words in italics expressing the very thing which had to be defined! The fact that such a measure of variable velocity has so long escaped logical destruction is marvellous.

say, in every 5 seconds the velocity of the point increases by 15 feet per second. In other words, we may say that there is a uniform rate of increase of velocity, at the rate of—

15 feet per second per 5 seconds.

If in *every* 5 seconds a velocity of 15 feet per second is added, we say that in every second a velocity of 3 feet per second is added.

Rate of increase of velocity per unit of time is called *acceleration*. In the above case there is a uniform acceleration which we may describe as—

an acceleration of 15 feet per second per 5 seconds, or
an acceleration of 3 feet per second per second.

It is extremely important to understand thoroughly the nature of *acceleration*, and to see that *time must be twice mentioned in it*. Thus, the expression "an acceleration of 32 feet per second" is wholly nonsensical; for 32 feet per second is a velocity, and the expression does not tell us in what time this velocity has been added—which every correct expression for acceleration should do.

What is the meaning of saying that a train is moving with a constant acceleration of 2 miles per hour per minute? The meaning is that the velocity of the train becomes 2 miles per hour greater every minute; so that if its velocity now is (say) 18 miles per hour, at the end of 1 minute more it will be 20 miles per hour, at the end of 2 minutes more it will be 22 miles per hour, at the end of 15 minutes from the present time it will be 48 miles per hour; and so on.

4. **A Particle.**—A very small portion of matter is called a *material particle*, or, simply, a *particle*. A particle, thus described, has, of course, no definite size. Sometimes we may consider a small marble a particle, and sometimes it might be necessary to consider a marble as a rather large mass containing many small particles. Similarly, for some purposes, a cricket-ball may be considered as a single particle of matter, and for other purposes it might be necessary to consider it as a collection of a vast number of small particles. Again, we can sometimes treat the Earth on which we live as a mere particle moving in a vast curve round the Sun.

5. **Force.**—There are several ways in which we obtain the notion of the thing which we call *force*. One way is this: We know that if we make a spiral spring of a piece of brass or steel wire, such as that represented in fig. 2, and we take hold of the ends A and B and pull them apart, the spiral lengthens, and we feel that exertion is necessary in order to stretch the spring. Each end of the stretched spring is pulling the hand in contact with it—or, as we commonly say, is exerting *force* on the hand. As long as the stretch of the spring remains constant, the force exerted by it is constant. An elastic cord of india-rubber would do as well as the metal spring.

Fig. 2.

Now imagine the spiral spring to be attached by the end A to a particle, M, of matter—placed (say) on a horizontal table—and the end B to be pulled by the hand, the spring being kept stretched, and, moreover, stretched to a constant amount. What will happen? Everyone will admit from experience that, in order to keep the spring at constant stretch, the hand at B must move faster and faster continually; in other words, if the stretch of the spring is kept constant, the particle will move faster and faster continually. No one has any difficulty in admitting this. But, what no one could say without experiment and measurement in this case is this: *If the stretch of the spring is kept constant, there is a constant acceleration produced in the motion of the particle.*

In other words, *constant force produces constant acceleration in the particle.*

This law is the foundation of all science: to understand it thoroughly is to conquer at once a large domain in Dynamics. It is contained in what is known as Newton's Second Axiom, or Law, of Motion.

If a particle moves in a straight line so that its velocity remains constant, it cannot be acted upon by any force; for if it were, its velocity would be accelerated; and, conversely, if the particle is not acted upon by any force, its velocity must remain constant. As regards the motion of a particle, then, we may briefly say: *no acceleration in any direction, no force in that direction*; and, conversely: *no force, no acceleration.*

6. **Representation of Velocities.** — Let us, for a moment, leave the consideration of *acceleration* and return to that of simple *velocity*. Velocity is a thing which has both *magnitude* and *direction*. If we are told, for example, that a particle is moving with a velocity of 20 feet per second, we are not, so far, told *everything* about its velocity; we require to be told the *direction* in which it takes place.

Let P (fig. 3) be the position of the particle at any instant, and let $A'PA$ be the right line in which P is moving at that instant; then if P is moving towards A, we represent the velocity by an arrow flying from P to A. If P is moving in the line $A'PA$, but towards A', we should represent the velocity by an arrow flying from P towards A'; so that, besides the right line $A'PA$ in which P is moving, we require to be told the *sense* along this line in which P is moving. Of course, we may include the *sense* in the *direction*; but it is not unusual to mean by *direction* merely the line $A'PA$ in which the velocity takes place, and if we do this, we shall require to be told the *sense* of the velocity in order that we may know whether the velocity takes place towards the right or towards the left of P. If we are told that a stone is moving in a vertical direction with a velocity of 100 feet per second, this, as commonly understood, would not enable us to say whether the stone is moving *upwards* or *downwards* with the velocity of 100. In this case we should distinguish the *upward sense* from the *downward sense*.

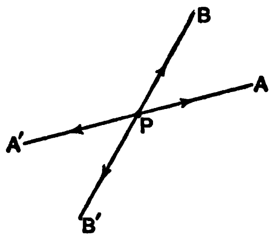


Fig. 3.

In this book we shall use *direction* to signify merely *straight line*, and we shall speak of velocities, accelerations, and forces as having, each of them, *magnitude*, *direction*, and *sense*.

The term *speed* is often used to denote the *magnitude* of a velocity without any reference to the line in which it takes place.

Now, in Dynamics we have to represent velocities, accelerations, and forces on paper. We cannot, in reality, *draw* a velocity, an acceleration, or a force on paper: all that we can do is to draw *representations* or pictures of these quantities; and it is very important to understand how this is done.

Suppose that the point P (fig. 4) has a velocity along the line PA, and in the *sense* PA, of n feet per second, and that we wish to represent this velocity. What we do is this: we take a straight line, LM, and take a series of equal divisions along it. Now each of these divisions may be taken to *represent* any speed we please. Suppose, for example, that the velocity of P along PA is 100 feet per second. Then, if we take each division along LM to represent a speed of 1 foot per second, the velocity along PA would be represented by a length equal to 100 of the divisions. This would be inconveniently long; so we take each division along LM to represent more than 1 foot per second. Say that each of the divisions represents a speed of 10 feet per second; then the given velocity of 100 feet per second is represented by a length Pa, which contains 10 of the divisions of LM—or 10 scale divisions, for LM is our *scale* of measurement.

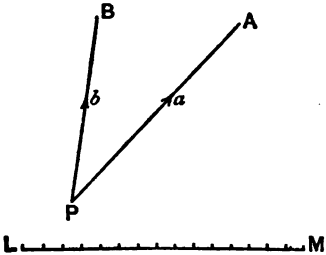


Fig. 4.

If the velocity is in the *sense* PA, we put an arrowhead at a on the line Pa, which represents P's velocity.

How should a velocity of 60 feet per second along PB and in the sense PB be represented *on the same scale*? Evidently by measuring a length, Pb, equal to 6 of the scale divisions and putting an arrowhead at b.

If each scale division on LM is taken to represent a speed of 20 feet per second, how will the two velocities, 100 along PA and 60 along PB, be represented? Pa and Pb should now be taken equal to 5 divisions and 3 divisions, respectively, of the scale.

The contraction $^f/_s$ is often used for the expression "feet per second"; thus, 100 feet per second is written $100^f/_s$. This contraction we shall use. If velocity is measured in miles per hour, we may use the contraction $^m/_h$; if in centimètres per second, the contraction $^c/_s$; and so on.

7. **Representation of Accelerations.**—To represent accelera-

tions we must, in the same way, represent a magnitude which has direction and sense; and, just as above, we must take some convenient length, such as one of the divisions of the line *LM* in fig. 4, to represent an acceleration of definite amount. For example, if we wish to represent an acceleration of 60 feet per sec. per sec. along the line *OA* (fig. 5), we may take each scale division to *LM* to represent an acceleration of $10^f/_{ss}$, and then measure off a length Oa on *OA* containing 6 scale divisions. We shall often in the sequel use a double-headed arrow, as in fig. 5, to indicate an acceleration, velocities being represented by single-headed arrows.

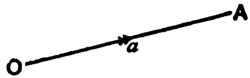


Fig. 5.

8. **Composition of Velocities.**—A particle may have two velocities along two different directions at the same time, as may be seen by the following illustration.

Suppose that *CDEF* (fig. 6) is a board in which there is a groove, *OG*, cut, the board lying on a table. Let there be in the groove a particle (such as a marble or a grain of shot) at *O*, and let this particle be moved along the groove with a velocity of (say) 5 centimètres per second at the same instant that the whole board—*i.e.* every point in the board—is moved along the table in the direction *OB* with a velocity of (say) 12 centimètres per second. Then at the end of 1 second where is the particle found? It must be found in the groove at a distance of 5 centimètres from the end *O*, but the groove at the end of a second is in the position *BG′*, parallel to *OG*, and such that *OB* = 12 centimètres. Hence if from *B* we draw *BA′* equal to 5 centimètres, the particle is at *A′*. We see, then, that if at *O* we draw *OA* = 5, and *OB* = 12 centimètres, the point *A′* is the extremity of the diagonal, *OA′*, of the parallelogram whose two adjacent sides are *OA* and *OB*. In the second, therefore, the

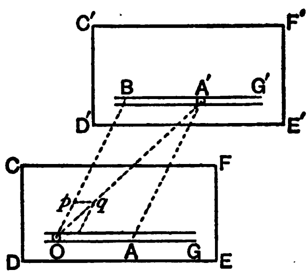


Fig. 6.

particle has described the distance OA'. Moreover, if we imagine the particle capable of dropping ink continuously down through the groove on the table beneath, the ink marks would make the right line OA' on the table. Why is this? Because if we seek for the position of the particle at the end of any fraction of a second, we shall find that it is on the line OA'. Thus, at the end of (say) $\frac{1}{4}$ of a second the end of the groove that was at O must be found at a distance $\frac{OB}{4}$ from O—*i.e.* at p if $Op=\frac{OB}{4}$,—and the particle must be found at a distance $\frac{OA}{4}$ in the groove from p; so that if we draw $pq=\frac{OA}{4}$, the particle is at q. But q lies on the line OA', because $\frac{Op}{pq}=\frac{OB}{OA}$; and, similarly, no matter what fraction of a second we take. We may therefore consider that the particle has a single velocity represented, both in magnitude and in direction, by the line OA'.

Generally, then, *if* OA *and* OB *represent velocities of any magnitudes and directions, a particle which has them both at the same instant, may be considered as having a single velocity represented in magnitude and in direction by the diagonal through* O *of the parallelogram whose adjacent sides are* OA *and* OB.

This result is known as the proposition of the *Parallelogram of Velocities*. For clearness we draw another figure (fig. 7) which represents this proposition. The point o is supposed to have a velocity, u, along one line, and at the same instant a velocity, v, along another, these being represented by the arrows in the figure; then these two are equivalent to a single velocity along the diagonal, or, of the parallelogram whose two adjacent sides are u and v. The single velocity, which is equivalent to two simultaneous velocities, is called the *resultant* of the two velocities.

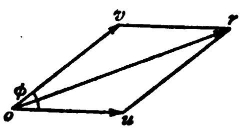


Fig. 7.

If the directions of u and v are at right angles to each other, their resultant, r, is given by the equation—

$$r = \sqrt{u^2 + v^2}.$$

If the directions of u and v include any angle ϕ, as in fig. 7, we have by (*Euclid*, Prop. xii., Book ii.)—

$$r = \sqrt{u^2 + 2uv\cos\phi + v^2}.$$

9. **Resolution of Velocities.**—It is evident that the single velocity r (fig. 7) can be replaced by the two simultaneous velocities u and v; and, generally, if a point O (fig. 8) has a velocity v represented by the arrow Ov, this velocity may be replaced by two velocities along any two right lines, OA and OB; for we have only to arrange that Ov is the diagonal of a parallelogram whose two adjacent sides lie along OA and OB. If from the point v we draw the lines vp and vq parallel to OB and OA, the parallelogram in question is $Opvq$, and the velocity Ov may be replaced by, or broken up into, the two simultaneous velocities Op and Oq. If we wish to replace Ov by velocities along OA and OC, we draw from v the lines vs and vt parallel to OC and OA, and we have the parallelogram $Osvt$, so that the velocity Ov can be broken up into the velocities Os and Ot.

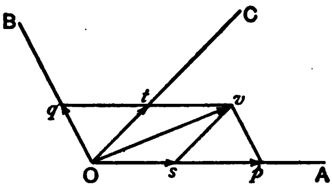


Fig. 8.

If a velocity is broken up into two velocities, these are called *components* of the given velocity. Thus Ot and Os are the components of Ov along the lines OC and OA; Oq and Op are the components of Ov along OB and OA.

A given velocity may be broken up, or *resolved* into, two velocities in an infinite number of ways, because an infinite number of parallelograms can be drawn having the same line for diagonal.

Is it possible to give a definite answer to the following question? *Given a velocity represented by* Ov, *what is its component along the line* OA? No; because you see above that Os may be the component of Ov along OA, and so may Op:

everything depends on the *other* line (OC, OB, . . .), along which the second component of Ov is taken.

We often, however, speak of *the* component of Ov along OA; but then we mean that Ov is resolved along OA and a perpendicular to OA, as in fig. 9. If OB is perpendicular to OA, the components of the velocity v, along and perpendicular to OA, are obtained by drawing the lines vp and vq parallel to OB and OA. The components of v (which is represented by Ov) are then represented by Op and Oq. If Ov makes the angle θ with OA, we have

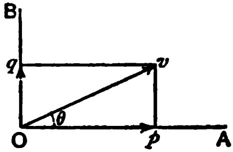


Fig. 9.

$$Op = v \,.\, \cos\theta, \text{ and } Oq = v \,.\, \sin\theta.$$

The component Op of Ov along OA when Ov is resolved along OA and a perpendicular, OB, to OA, is called the *rectangular component* of Ov along OA. As rectangular resolutions are the simplest, the word "rectangular" is omitted but always understood when we speak of the *component of a velocity along a given line.*

Thus, then, we say that—

the component of a velocity, v, *along any right line is* v . cos θ, *where* θ *is the acute angle between* v *and the line;*

and by "component" here we mean "rectangular component" —*i.e.* the velocity is supposed to be resolved along the given line and along a perpendicular to it.

It is most important for the beginner to train his eye in such a way that he can tell at a glance the *sense* in which the component of a given velocity along a given line takes place. Thus, in the following figure (fig. 10)—

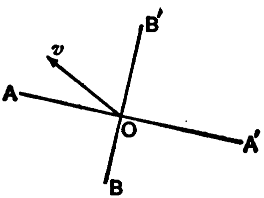


Fig. 10.

what is the sense of the component of v along the line AA'?

what is the sense of the component of v along the line BB'?

The foot of the perpendicular let fall from the extremity of v on the given line will always show the sense of the component.

10. **The Triangle of Velocities.**—A triangle may be used instead of a parallelogram for resolving a given velocity into two components. Thus, suppose that it is required to resolve a velocity of $25^f/_s$ into velocities of $18^f/_s$ and $10^f/_s$: draw the line *Ov* (fig. 11) to represent $25^f/_s$ an any scale, and on *Ov* as base describe a triangle, *Opv*, whose two remaining sides are proportional, on the same scale to 18 and 10; then the directions and senses of the two component velocities are represented by the arrows *Op* and *pv*. For, if from *v* and *O* we draw *vq* and *Oq* parallel to the sides 18 and 15 of the triangle *Opv*, we have the parallelogram *Opvq*, whose two sides, *Op* and *Oq* represent the components 18 and 10 into which the velocity *Ov* can be resolved, *Ov* being the diagonal of the parallelogram.

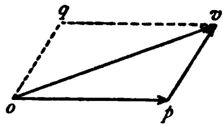


Fig. 11.

A velocity cannot be resolved into two given components unless the sum of these components is greater than the given velocity, because no triangle can be constructed with three given lines unless the sum of each two is greater than the third.

EXERCISES

1. How does a velocity of 60 miles per hour compare with a velocity of 88 feet per second?

To express $60^m/_h$ in feet per second, observe that in a mile there are 5280 feet, and in an hour there are 3600 seconds, so that—

$$60 \text{ miles per hour} = 60 \times 5280 \text{ feet per } 3600 \text{ seconds}$$
$$= \frac{60 \times 5280}{3600} \text{ feet per 1 second.}$$
$$= 88^f/_s,$$

so that the two velocities are the same.

2. Express a velocity of 12 miles per hour in yards per minute.
Ans. 352 yards per minute.

3. Express a velocity of 72 mètres per hour in centimètres per second.
Ans. $2\ ^c/_s$.

4. Express an acceleration of 32 feet per second per second in miles per hour per second.

$$\begin{aligned}
&32 \text{ feet per sec. per sec.}\\
&=\tfrac{32}{5280} \text{ miles per sec. per sec.}\\
&=\tfrac{32}{5280} \text{ miles per } \frac{\text{hour}}{3600} \text{ per sec.}\\
&=\frac{32\times 3600}{5280} \text{ miles per hour per sec.}\\
&=\tfrac{240}{11} \text{ miles per hour per sec.;}
\end{aligned}$$

that is to say, the velocity of a particle which has the given acceleration will increase by $21\frac{9}{11}$ miles per hour at the end of every second; or, in other words, if the velocity is (say) 5 miles per hour at the present moment, then it will be—

$26\frac{9}{11}$, $48\frac{7}{11}$, $70\frac{5}{11}$, $92\frac{3}{11}$, . . . miles per hour

at the end of—

1, 2, 3, 4, . . . seconds from the present moment.

5. If a train is moving with an acceleration of $\frac{22}{15}$ inches per second per second, what is its acceleration in miles per hour per minute?

$$\begin{aligned}
&\tfrac{22}{15} \text{ inches per second per second}\\
&=\frac{22}{15\times 12\times 5280} \text{ miles per } \frac{\text{hour}}{3600} \text{ per } \frac{\text{minute}}{60}\\
&=\frac{22\times 3600}{15\times 12\times 5280} \text{ miles per hour per } \frac{\text{minute}}{60}\\
&=\frac{22\times 3600\times 60}{15\times 12\times 5280} \text{ miles per hour per minute}\\
&=5 \text{ miles per hour per minute;}
\end{aligned}$$

that is, if its velocity at the present moment is (say) 6 miles per hour, it will be—

11, 16, 21, 26, 31, . . . miles per hour

at the end of—

1, 2, 3, 4, 5, . . . minutes

from the present moment.

6. Express an acceleration of 5 miles per hour per quarter-hour in inches per second per second.

Result. $\frac{22}{225}$ ″/ss.

7. If a velocity of 34 feet per second is resolved into two components, 30 f/s and 16 f/s, what is the angle between the directions of these components?

Ans. A right angle.

8. Two velocities, u and v centimètres per second, in directions at right angles to each other, have a resultant of 13 $^c/_s$; if each of the components is increased by 3 $^c/_s$, the resultant becomes 17 $^c/_s$; find u and v.

Result. 5 and 12 $^c/_s$.

9. Resolve a velocity of 52 $^f/_s$ into two components at right angles to each other, these components being in the ratio 5 : 12.

Result. The components are 20 and 48 $^f/_s$.

10. If a velocity of 15 is resolved into one of 14 and one of 13, what is the angle between the 15 and the 14, and the angle between the 14 and the 13?

Ans. $\cos^{-1} \frac{3}{5}$ and $180° - \cos^{-1} \frac{5}{13}$, respectively.

EXAMINATION ON CHAPTER I

1. What are the fundamental unit quantities in Dynamics?
2. What is a mètre? How does it compare with the English yard?
3. At what temperature is water most dense?
4. Define a gramme mass.
5. What is the definition of *velocity*?
6. What is *acceleration*? Is there such a thing as an acceleration of 5 miles per minute?
7. If a constant force of any magnitude acts on a given particle in a fixed right line, what will be the kind of motion produced in the particle?
8. If a particle is observed to be moving in a fixed right line with invariable velocity, what do we know about *force* in connection with this motion?
9. If a particle is observed to be moving in a fixed right line with variable velocity but constant acceleration, what do we know about *force* in connection with this motion?
10. How are the magnitude and the direction of a given velocity represented? What is meant by a *scale* of velocities?
11. What is meant by the *sense* of a velocity along a given right line?
12. Give any instance in which a particle may be said to have two velocities, at the same time, along two different directions?
13. State the proposition of the *Parallelogram of Velocities*.
14. Can a velocity of 20 feet per second be broken up into two velocities of 12 $^f/_s$ and 15 $^f/_s$? Can it be broken up into velocities of 12 $^f/_s$ and 5 $^f/_s$?
15. Is the expression "the component of a given velocity along a given direction" perfectly definite? What is commonly understood by "the component of a given velocity along a given direction"?
16. What is the expression for the resultant of two given velocities?

CHAPTER II

FORCE (NEWTON'S SECOND AXIOM, OR LAW, OF MOTION)

11. **Definition and Measurement of Force.**—Whatever causes the motion of a material particle to be accelerated is called *force*. Many different units of force may be taken. One particular unit of force is thus defined: Imagine a mass of 1 gramme (fig. 12) placed on a perfectly smooth horizontal table; to it let a very delicate spiral spring (or elastic cord) be attached, and stretched to such an extent that, if the stretch is kept constant while the gramme is moved along by the spring, there will be produced an acceleration of 1 centimètre per second per second, denoted by $1^c/_{ss}$ in the figure; then the force which the spring exerts is called a *dyne*. A *dyne*, then, *is the force which produces an acceleration of* $1^c/_{ss}$ *in a gramme mass*. This is called an *absolute unit* of force, for a reason which will be given later on. If another spiral spring, identical in every way with the first, and stretched to the same extent, is, in addition, applied to the gramme, so that the gramme is acted upon by 2 dynes instead of 1 dyne, we assume that the acceleration will be $2^c/_{ss}$; and if any number, n, of spiral springs, each producing a force of 1 dyne, be attached to a mass of 1 gramme, the acceleration produced will be $n\,^c/_{ss}$.

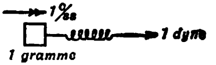


Fig. 12.

How many dynes must act on 1 gramme mass to produce an acceleration $a\,^c/_{ss}$? Obviously a.

How many dynes will be required to produce an acceleration of $1\,^c/_{ss}$ in a mass of m grammes? It is easy to see that m dynes will be required, thus: Imagine m grammes placed side by side, as in fig. 13, and each acted upon by a force of 1 dyne; then each will have an acceleration of $1\,^c/_{ss}$. Now, imagine the m grammes all glued, or otherwise united, together,

and we shall have a compound body of *m* grammes mass acted upon by *m* dynes, and having, in consequence, an acceleration of 1 $^c/_{ss}$.

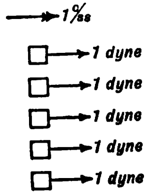


Fig. 13.

How many dynes must act on a mass of 5 grammes to produce an acceleration of 12 $^c/_{ss}$? It is not difficult to see now that we require 5×12, or 60, dynes. For, imagine 5 separate gramme masses placed side by side, as in fig. 13, and let an acceleration of 12 $^c/_{ss}$ be produced in each. How many dynes will be required for each mass? By what we assumed above, each mass will require 12 dynes. Now, if the 5 grammes are imagined as united into one body, this mass of 5 grammes will have the acceleration 12 $^c/_{ss}$, and the force applied to it is 60 dynes.

We can now see that if a force applied to a mass of *m* grammes produces in the mass an acceleration of a $^c/_{ss}$, that force contains $m \times a$ dynes.

If, then, a force of *P* dynes is applied to a mass of *m* grammes, and produces an acceleration of a centimètres per second per second, we have the result—

$$P = m.a \quad . \quad . \quad . \quad . \qquad (1)$$

If instead of a gramme mass we take a mass of one *pound*, and pull the spiral spring in such a way that the acceleration produced is 1 *foot per second per second*, the force exerted by the spring is called a *poundal*; so that if a force of P poundals acts on a mass of *m* pounds, the acceleration, a $^f/_{ss}$, which it produces is given by (1).

Any two forces may be compared with each other by means of the accelerations which they produce in the same mass, no matter what that mass is.

Suppose that we take the same mass, *m* grammes, in which one force, *P*, produces an acceleration a $^c/_{ss}$, and another force, *Q*, produces an acceleration β $^c/_{ss}$; then we have—

$$Q = m.\beta \quad . \quad . \quad . \quad . \qquad (2)$$

for the number of dynes in *Q*. Now from (1) and (2) we see that—

$$\frac{P}{Q} = \frac{a}{\beta} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

which shows that the forces are in the ratio of the accelerations

which they produce in the *same* mass, and it does not matter how many grammes that mass contains.

This is an extremely important result, and it allows us to measure force magnitudes in many ways. For, supposing that any force F acting on a given body produces an acceleration f in the body, and that α is the acceleration produced in the body by a force P, we have—

$$P = F.\frac{\alpha}{f} \quad . \quad . \quad . \quad . \quad (4)$$

and if a force Q produces an acceleration β in the same body,

$$Q = F.\frac{\beta}{f};$$

so that all forces may be expressed as multiples of the force F, which we may regard as a kind of *standard force* in terms of which to express all other forces.

Observe that when we express a force acting on a mass m in the form (1), *we are not expressing it as a multiple of any other force acting on the body.* Of course (1) expresses P as a multiple of the force called a *dyne*; but this dyne is not a force specially related to the given body m.

For any given body on the Earth's surface, what is the most convenient force to adopt as a *standard force*, in terms of which to express all the other forces that may act on the body? *It must be such a force that the acceleration which it produces in the body is accurately known;* and such a force is that which we call the *weight* of the body.

12. **Nature of Weight.**—The Earth is, roughly speaking, a sphere 8000 miles in diameter, not of the same density all through (*i.e.* not having the same quantity of matter in a cubic foot all through), but divisible into spherical layers whose densities increase towards the centre of the Earth. Such a body attracts any other body outside it as if it (the Earth) were condensed into a single particle at its centre. The force, or pull, which the Earth exerts on any *outside* mass m (fig. 14) can be proved to vary inversely as the square of the distance of m from the centre, O, of the Earth. Supposing the mass m to be placed succes-

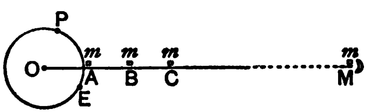


Fig. 14.

sively at the surface at *A*, at *B*, at *C*, . . . at *M*, such that $OA = AB = BC = \ldots$, the pull exerted on the mass *m* at *B* is $\frac{1}{4}$ of the pull exerted on it when it was at *A*, since $OB = 2.OA$, and the force varies inversely as the square of the distance from *O*. The pull exerted on the mass *m*, if placed at *C*, will be $\frac{1}{9}$ of the pull exerted on it at *A*; and so on.

If *M* is the position of the Moon, *OM* is about 60 times the Earth's radius, so that the pull exerted by the Earth on the mass *m* when *m* is at the distance of the Moon is—

$$\frac{1}{3600}$$

of the pull exerted on the same mass when placed on the Earth's surface.

If *m* could be carried inside the Earth from *A* towards *O*, it would be found that the pull exerted by the Earth on it towards *O* gets less and less, and that when *m* reaches *O*, the Earth exerts no force at all on it. The greatest force, or pull, exerted on *m* occurs *at the surface*; and the force diminishes if *m* is moved from *A* either towards *O* or away from it.

The particular value of the force exerted by the Earth on the body *m* when *m* is at the Earth's surface is called the *weight* of the body *m*. This force will remain *nearly* of the same amount no matter at what point, *E*, *A*, *P*, . . ., of the Earth's surface the mass *m* is placed. Since the Earth is not quite spherical, but somewhat flattened at the pole *P*, the force exerted on *m* is a little greater when *m* is placed at *P* than when it is placed at the Equator, at *E*.

Hence the weight of one and the same mass *m* varies a little with the place of *m* on the surface. Now, observe particularly that in all the changes of position of *m* from *O* to *A*, *B*, *C*, . . . *M*, . . ., there is one thing which is always the same—namely, the *mass of the body*. The body is composed of exactly the same particles at *M* as at *A*; but the pull of the earth on it has been extremely variable. At the position *M* it is very small; at a distance of a million miles from *O* it would be practically nothing at all.

The *mass* of a body, or quantity of matter in it, is therefore quite distinct from the *weight* of the body, this latter being merely a very particular force exerted upon it in a very particular position—the force exerted upon it by the attraction of a particular planet, called the Earth, when the mass is on the surface of the planet. If the very same body, *m*, were

taken to the surface of the planet Jupiter, its "weight" (*i.e.* the pull exerted on it by Jupiter) would be, roughly, $2\frac{1}{2}$ times its "weight" on the surface of the Earth. If the body were placed near the surface of the Sun, its "weight" would be about 28 times its "weight" here; if it were placed near the surface of the star Arcturus, its "weight" would be (there is some reason for believing) nearly 90 times what it would be at the surface of the Sun; *i.e.* more than 2500 times its value on the surface of the Earth.

The variation of the weight of a body at different places on the Earth's surface might be shown by means of a delicate spring-balance. If a spiral spring has attached to it a hand and a dial, the face of which is divided into a large number of equal parts, and a given mass is attached to the spring, it will cause the hand to move from the zero position along the face of the dial. Let this spring be used with a suspended mass at the equator, and suppose the hand to move over 100 divisions. Then if the same mass is suspended from the same spring at the pole, the hand will move over $100\frac{1}{2}$ divisions, very nearly.

13. **Acceleration Produced in a Body by its Weight.**—If a material particle is allowed to fall freely at the Earth's surface—say from the ceiling of a room to the floor, or from the top of a tree to the ground—it is, during the whole motion, pulled by the Earth with a force which we call its weight, this being an almost perfectly constant force, since we may suppose the ratio of the distance of the falling body from the Earth's centre to the radius of the Earth to be equal to unity in all positions of the body. Constant force on the body produces constant acceleration; and it is found in various ways by measurement that the acceleration here is about—

32 feet per second per second,

or about—

981 centimètres per second per second.

This acceleration produced in the motion of a freely-falling body is always denoted by the symbol—

$$g.$$

How many poundals are there in the weight of a body?

If the body contains m pounds or W pounds—it does not matter which letter we use—the number of poundals in the force called its weight is about $32\ m$ or $32\ W$. Thus, if the body contains 3 pounds of matter, its weight contains 96 poundals; and a poundal is about the weight of half-an-ounce.

Of course, the weight of the body *estimated in pounds' weight* is m, or W.

14. **Absolute Measure and Gravitation Measure of a Force.**—Having assumed a unit of mass and a unit of acceleration, if we define a unit of force to be that force which will produce the unit of acceleration in the unit of mass, this unit of force is called an *absolute unit of force*, because it would be the same, not only at all points on the surface of the Earth, but at all points on every planet and everywhere in space. We can easily imagine other ways of taking an absolute unit of force. For example, we may rely on *elasticity* to give us such a unit. We may say that a steel spring of given dimensions, and at a certain temperature, when it is stretched to any amount (such as $\frac{1}{100}$ or $\frac{1}{1000}$ of its natural length) is exerting a unit force; and we could express every force as a multiple of this unit. Such a unit force would, like the dyne or the poundal, be the same everywhere on the Earth and everywhere in space; but its definition is not so simple as that of the dyne or the poundal.

If we take the *weight* of some chosen mass, such as a pound or a ton mass, as the unit of force, we can, of course, express the magnitude of every force as a multiple of this unit. But the weight of a given mass is not the same all over the Earth, and has no definite meaning at all when the mass is out in space remote from the Earth. For example, the force which would break a given piece of elastic cord on the surface of the Earth is precisely the same as the force which would break it on the surface of Jupiter. If an inhabitant of the earth found that the breaking force was ten times the weight of a pound mass—or, as we commonly say, 10 pounds' weight—an inhabitant of Jupiter would find that the breaking force is about 4 pounds' weight, because the weight of a pound on Jupiter is about $2\frac{1}{2}$ times the weight of the same mass on the earth.

When a force is expressed in terms of the *weight* of some unit mass, it is said to be expressed in *gravitation measure*;

if it is expressed in terms of an absolute unit of force, such as a dyne or a poundal, it is said to be expressed in *absolute measure.*

This will be understood by considering the question: What is the magnitude of a force, P, which produces an acceleration of 20 feet per second per second in a mass of 8 pounds? By what has been already said, the force contains—

$$8 \times 20, \text{ or } 160, \text{ poundals.}$$

If we please, we may (see Art. 11) compare P with the weight of the body—*viz.* 8 pounds' weight; and then we have by (4), Art. 11—

$$P = 8 . \tfrac{20}{32} = 5 \text{ pounds' weight.}$$

Generally, then, if a force, P, acting on a particle of w pounds, produces an acceleration of a feet per second per second, we have—

$$P = w.a \text{ poundals, } \quad . \quad . \quad . \quad . \qquad (1)$$

or

$$P = w\frac{a}{g} \text{ pounds' weight, } \quad . \quad . \qquad (2)$$

the first equation giving the value of P in British *absolute measure*, and the second in *gravitation* measure.

If the mass of the particle, w, is measured in grammes, and the acceleration, a, in centimètres per second per second, we have—

$$P = w.a \text{ dynes, } \quad . \quad . \quad . \quad . \qquad (3)$$

or

$$P = w\frac{a}{g} \text{ grammes' weight, } \quad . \quad . \qquad (4)$$

g being now about 981 centimètres per second per second.

Engineers almost invariably use the gravitation measure of forces—such as pounds' weight or tons' weight. The British Absolute Measure (poundals) is not used; but the C.G.S. (*i.e.* centimètre-gramme-second) Absolute Measure is very largely used in all calculations relating to electricity and magnetism.

15. **Pound and Gramme.**—It will be useful to note the fact that—

$$1 \text{ pound} = 453.59 \text{ grammes, nearly.}$$

EXERCISES

1. Express a pound weight in dynes.
   Result. 445×10^3 dynes, nearly.

2. A certain force is capable of producing an acceleration of 6 $^f/_{ss}$ in a mass of 8 pounds, what acceleration can it produce in a mass of 3 pounds, and how many pounds' weight are there in the magnitude of the force?
   Result. 16 $^f/_{ss}$; $1\frac{1}{2}$ pounds' weight (taking g as 32 $^f/_{ss}$).

3. What acceleration can a force of a million dynes produce in 1 pound mass?
   Ans. 2204.6 $^c/_{ss}$, nearly.

4. Express in pounds' weight the force which can produce an acceleration of 8 inches per second per second in a mass of 4 ounces.
   Result. $\frac{1}{192}$ pounds' weight.

5. A certain force produces in a certain mass an acceleration of 10 $^f/_{ss}$; if 1 pound is taken from the mass, the same force produces in the remaining mass an acceleration of 12 $^f/_{ss}$; find the force and the mass.
   Result. The mass is 6 pounds, and the force is $1\frac{7}{8}$ pounds' weight.

EXAMINATION ON CHAPTER II

1. What is the definition of *force*?
2. If a *constant* force acts on a particle, what is the nature of the particle's motion?
3. Describe any way in which a constant force may be produced.
4. Define a *dyne*. What fraction of a gramme weight is it? ($\frac{1}{981}$.)
5. How can the magnitudes of all forces be compared with each other by means of their effects on one and the same particle?
6. What is meant by the *weight* of a body? Is its *weight* the same as its *mass*?
7. Where on the Earth's surface would a given mass (say a cricket ball) have the greatest weight? How could the change of weight of a given mass be shown by a spring balance?
8. What would happen to the weight of a cricket ball if the ball could be taken through the Earth towards its centre?
9. What is the length of the Earth's radius? With what force would the Earth pull a cricket ball, if the ball were placed at a distance of 2000 miles from the Earth's surface. [*Ans.* $\frac{4}{9}$ of its weight at the surface.]
10. How many times heavier would a given particle be if it were placed at the surface of the Sun (and not destroyed), than it would be at the surface of the Earth?
11. What acceleration is produced in a body when it falls freely near the Earth's surface? [Give the answer in $^f/_{ss}$ and in $^c/_{ss}$.]
12. If a body has a mass of w pounds, can we say that its weight is w? Can we say that its weight is $32\,w$?
13. What is meant by an *absolute unit of force*?
14. What is the weight, in dynes, of a mass of 5 grammes? What is its weight in grammes' weight?
15. What is the *gravitation measure* of a force?

CHAPTER III

THE COMPOSITION AND RESOLUTION OF FORCES (NEWTON'S SECOND AXIOM, OR LAW, OF MOTION)

16. **Resultant of Two Forces.**—If a particle is acted upon by two forces at once, there is a single force which will produce the same effect on the particle as the two forces acting together. This result follows at once from our definition and measure of force. For, let a particle, O (fig. 15), be acted upon by a force, P, in the line OA, and a force, Q, in the line OB; then let P and Q be measured by the accelerations which they separately produce in the particle; that is, P may be measured by the velocity which it gives the particle at the end of a second (or any other unit of time) if the particle started from rest, and Q may be measured in the same way. Draw, then, OP to represent the velocity thus produced in the particle by the force, P, and OQ to represent, on the same scale, the velocity produced by Q. Now we know that if O has the velocities OP and OQ at the same time, it will have the single, or resultant, velocity represented by OR, the diagonal of the parallelogram whose adjacent sides are OP and OQ (Art. 8).

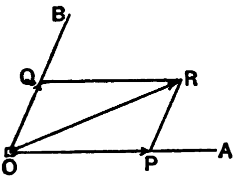


Fig. 15.

Hence OR represents, on the same scale, a single force which would produce on the particle the same effect as is produced by the two combined forces P and Q.

Also if ϕ is the angle QOP between the lines of action of P and Q, and if we denote their *resultant* by R, we have—

$$R^2 = P^2 + 2PQ\cos\phi + Q^2. \quad . \quad . \quad . \quad (1)$$

If the angle between the lines of action of P and Q is a right angle (fig. 16)—

$$R^2 = P^2 + Q^2; \quad . \quad . \quad . \quad (2)$$

and in this case, if R makes the angle θ with P,

$$\tan\theta = \frac{Q}{P}. \quad . \quad . \quad . \quad . \quad (3)$$

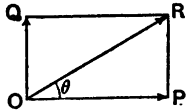


Fig. 16.

If the two forces P and Q act in the same line, then in (1) the angle ϕ is 0 or π, according as PQ are in the same sense or in opposite senses, and we have—

$$R = P \pm Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We see, then, that two forces are compounded into one exactly as two velocities are—*viz.* by the *parallelogram law*; and we may formally enunciate the proposition of the *parallelogram of forces* as follows: *If two forces, acting at the same point* O, *are represented in magnitudes, directions, and senses by two right lines drawn through* O, *their resultant is similarly represented by the diagonal, through* O, *of the parallelogram determined by these two lines.*

COR.—*The resultant of two equal forces bisects the angle between them*; for the diagonal of a parallelogram whose adjacent sides are equal bisects the angle.

Observe that we may speak of two or more forces as acting *at* a point if their lines of action pass through the point; but we should never speak of forces as acting *on* a point: force can act upon *matter* only—not on a point.

It is now obvious that *any* number of forces acting at a point (or on a particle) can be replaced by one force; for, taking any two of them, we can replace them by one; taking this one with another of the forces, we can again replace these two by one, and so on, until we are left with *one* force, which is the resultant, or complete equivalent, of all the given forces.

17. **Newton's Second Axiom, or Law, of Motion.**—*If a particle is acted upon by any number of forces, its resultant acceleration coincides at each instant with the resultant force acting upon it, and the product of the mass of the particle and its resultant acceleration is the absolute measure of the resultant force.*

Thus, if R (fig. 17) denotes the resultant force acting on a particle of mass w, then the resultant acceleration, a, of the particle will take place along, and in the sense of, R, and we have—

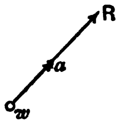


Fig. 17.

$$R = w.a, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or—

$$R = w.\frac{a}{g}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

according as we measure R in absolute or in gravitation units, as already explained in p. 22.

COR.—The *component* of a in any direction is, similarly, the equivalent of the component of R in that direction; that is, if a_x is the acceleration of the particle in any direction, and X the total component force acting on the particle in that direction (fig. 18), we have—

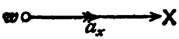


Fig. 18.

$$X = w.a_x, \quad . \quad . \quad . \quad . \quad (3)$$

or—

$$X = w.\frac{a_x}{g}, \quad . \quad . \quad . \quad . \quad (4)$$

according as force is estimated in absolute or in gravitation measure.

The law at the head of this article is the foundation of Dynamics—indeed, it is the *whole* of Dynamics, so far as the motion of a single particle is concerned. We shall refer to it as Newton's Second Axiom, although Newton's enunciation is in different words. In the "Principia" it stands thus: *Change of motion is always proportional to the impressed force, and takes place in the direction of the right line in which that force is impressed.*

But these words require much amplification to make their meaning clear. Thus, the expression, "change of motion," must be understood as meaning "time-rate of change of momentum." We shall not, however, dilate further on this at present; nor shall we require any other conception of Newton's Second Axiom than that contained in the words at the head of this article.

EXERCISES

1. Two forces of 16 and 30 pounds' weight act at right angles on a particle; what is the magnitude of their resultant, and what angle does its line of action make with the force 30?

Ans. The magnitude is $\sqrt{16^2+30^2}=2\sqrt{8^2+15^2}=34$ pounds' weight. It makes $\tan^{-1}\frac{8}{15}$ with the 30.

2. If the two forces in the last example act at right angles on a mass of 2 ounces, what acceleration, in magnitude and direction, will they produce in it?

Their resultant is 34 pounds' weight, and the weight of the particle being $\frac{1}{8}$ pound weight, we have from the equation—

$$P=w\frac{a}{g},\quad \text{(equation (2), p. 22)}$$

$$34=\tfrac{1}{8}\cdot\frac{a}{32},$$

$$\therefore\ a=8704 \text{ feet per sec. per sec.},$$

and a takes place in a line making $\tan^{-1}\frac{8}{15}$ with the force 30.

3. Two forces of 35 and 12 pounds' weight act at right angles on a mass of 8 pounds, what acceleration do they produce in it?

Ans. 148 $^f/_{ss}$ in a line making $\tan^{-1}\frac{12}{35}$ with the force 35.

4. Two forces of 20 and 21 grammes' weight act at right angles on a mass of 1 kilogramme (*i.e.* 1000 grammes) what acceleration do they produce in it?

Ans. 28.449 $^c/_{ss}$. [The resultant is 29 grammes' weight, and if it produces a $^c/_{ss}$ in a mass of 1000 grammes, we have $29=1000\frac{a}{981}$, $\therefore\ a=28.449$ $^c/_{ss}$.]

The direction of the acceleration makes $\tan^{-1}\frac{21}{20}$ with the force 20.

5. Two forces of 20 and 21 dynes act at right angles on a mass of 5 grammes, what acceleration do they produce?

Ans. $\frac{29}{5}$ $^c/_{ss}$. [The forces being now in *absolute measure*, we use the equation (3), p. 22.]

18. **Resolution of Forces.**—The composition and resolution of forces follow precisely the same rules as the composition and resolution of velocities, because (Art. 11) *all forces can be completely represented by the magnitudes and directions of the velocities which they would generate in the same time in the same particle.*

Hence, as in fig. 15, a force represented by a given right line can be replaced by two forces, represented by the adjacent sides of any parallelogram which has the given line for diagonal.

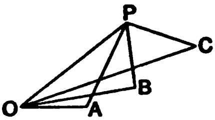


Fig. 19.

And, of course, we have also the result that any force, suppose OP, fig. 19, can be resolved into two forces represented by the sides of any *triangle* which has OP for base. Thus the force OP at O can be resolved into a force represented by OA and another represented in magnitude and sense by AP, but this latter must be placed at O parallel to AP. Similarly OP can be resolved into OB, and one at O parallel to BP; or into OC, and one at O parallel to CP; and so on.

Rectangular resolutions are most frequently employed; *i.e.* the force OP (fig. 20) is resolved into two, whose lines of

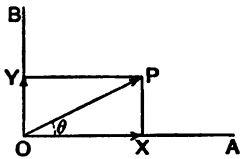


Fig. 20.

action are at right angles to each other. If OA is any direction, and OB the perpendicular direction, the components of OP along these lines are obtained by drawing from P perpendiculars, PX and PY, to OA and OB; then the force OP could be replaced by the two OX and OY. If OP makes the angle θ with OA, and if the magnitude of the force OP is denoted by P, while that of OX is denoted by X, and that of OY by Y, we have—

$$X = P.\cos\theta \quad . \quad . \quad . \quad . \qquad (1)$$

$$Y = P.\sin\theta \quad . \quad . \quad . \quad . \qquad (2)$$

As in the case of a velocity (Art. 9), the force P could be resolved in such a manner as to have an infinite number of different components along one and the same line, OA. These components depend on the position of the second line, OB, along which the other force acts. We shall call OX in fig. 20 *the* component of OP along OA, because we shall assume that we mean the *rectangular* component.

Thus, then, by the expression "the component of a force along a line," we shall mean the rectangular component, and (1) shows that—

The component of a force along a line = the force multiplied by the cosine of the acute angle between the force and the line.

It is extremely important that the student should, *at a glance*, recognise the sense in which the component of a given force along a given line acts: *the acute angle always determines this sense.* Thus, take fig. 21:

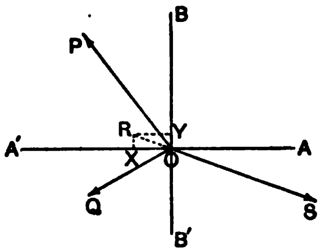


Fig. 21.

Let OA and OB be any two lines, and P, Q, R any forces acting at O.

What is the sense of the component of P along OA? From O towards A', because the foot of the perpendicular from the point P on OA will fall on OA'. What is the sense of the component of S along OB? From O towards B', and this component is $S \times \cos SOB'$; and so on. [The student should practise these resolutions in the above figure and similar ones.]

Understanding that we are speaking of rectangular components, *a force has no component at right angles to its own direction.*

19. **Resultant of any number of Forces.**—There are two ways in which the resultant of any number of forces acting on a particle may be found. We shall first explain one which depends on the resolution of forces; and a numerical example, to begin with, will make the process clear.

Suppose that O (fig. 21), is a particle acted upon by the three forces P, Q, S, whose magnitudes are respectively, 50, 34, and 52 pounds' weight, and with a given line, $A'A$,

$$\begin{array}{lll} \text{let } P \text{ make} & \tan^{-1} \tfrac{4}{3}, \\ \;\;,, \;\; Q \;\;\;,, & \tan^{-1} \tfrac{8}{15}, \\ \;\;,, \;\; S \;\;\;,, & \tan^{-1} \tfrac{5}{12}. \end{array}$$

Draw the line $B'B$ perpendicular to $A'A$, and resolve each

of the forces into two components, one along $A'A$ and the other along $B'B$.

Along $A'A$—

P gives a component $50 \times \frac{3}{5}$, or 30 in the sense OA',
Q „ „ $34 \times \frac{15}{17}$, or 30 „ OA',
S „ „ $52 \times \frac{12}{13}$, or 48 „ OA.

Thus we have a total component force of $30 + 30 - 48$ acting from O towards A'—*i.e.* 12 in OA'.

Now collect the components along BB'. The cosine of the angle which P makes with OB in $\frac{4}{5}$; and we see that along BB'—

P gives a component $50 \times \frac{4}{5}$, or 40 in the sense OB,
Q „ „ $34 \times \frac{8}{17}$, or 16 „ OB',
S „ „ $52 \times \frac{5}{13}$, or 20 „ OB';

so that in the sense OB we have the total, $40 - 16 - 20$—*i.e.* 4 pounds' weight.

Hence our given forces are thus reduced to two rectangular components—*viz.* 12 from O towards A' and 4 from O towards B; and the resultant of these two components is the resultant of the originally-given forces. To find the resultant, therefore, we lay off OX to represent 12 pounds' weight, and OY to represent, on the same scale, 4 pounds' weight; then the diagonal, OR, of the rectangle determined by OX and OY represents the resultant completely—*i.e.* in magnitude, line of action, and sense. Denoting the resultant by R, we have—

$$R^2 = 12^2 + 4^2 = 4^2 \times 10,$$

$$\therefore R = 2\sqrt{10} \text{ pounds' weight;}$$

and if θ is the angle made by R with OA',

$$\tan\theta = \tfrac{1}{3}.$$

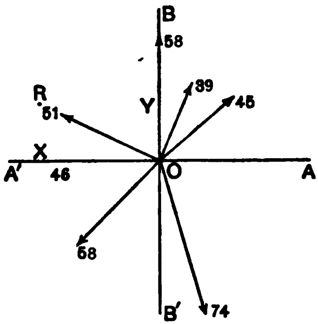


Fig. 22.

As another example, find the resultant of the following forces acting on a particle at O (fig. 22):

46 pounds' wt. in OA'			
51	,,	making	$\tan^{-1} \frac{8}{15}$ with OA'
58	,,	,,	a right angle with OA'
39	,,	,,	$\tan^{-1} \frac{12}{5}$ with OA
45	,,	,,	$\tan^{-1} \frac{3}{4}$,, ,,
74	,,	,,	$\tan^{-1} \frac{35}{12}$,, ,,
58	,,	,,	$\tan^{-1} \frac{21}{20}$,, OA'

Let us select the line $A'A$ and its perpendicular $B'B$ as those along which to resolve every force.

Denoting by X the total component along OA and in the sense OA, and by Y the total component along OB and in the sense OB, we have—

$$\begin{aligned} X &= 45 \times \tfrac{4}{5} + 39 \times \tfrac{5}{13} - 51 \times \tfrac{15}{17} - 46 - 58 \times \tfrac{20}{29} + 74 \times \tfrac{12}{37} \\ &= 36 + 15 - 45 - 46 - 40 + 24 \\ &= -56; \\ Y &= 58 + 39 \times \tfrac{12}{13} + 45 \times \tfrac{3}{5} - 74 \times \tfrac{35}{37} - 58 \times \tfrac{21}{29} + 51 \times \tfrac{8}{17} \\ &= 58 + 36 + 27 - 70 - 42 + 24 \\ &= 33. \end{aligned}$$

Hence the given forces are reducible to a force of 56 pounds' weight along OA' and in the sense OA', together with a force of 33 pounds' weight along OB; and the resultant, R, is therefore obtained by drawing OX along OA' to represent 56 pounds' weight, and OY along OB to represent 33 pounds' weight, and then drawing the diagonal OR of the rectangle determined by OX and OY. We have, then—

$$R^2 = 56^2 + 33^2$$
$$\therefore R = 65 \text{ pounds' weight,}$$
$$\tan\theta = \tfrac{33}{56},$$

θ being the angle ROA'.

If the body on which the forces act at O has a mass of 5 ounces, what are the magnitude and direction of the acceleration produced in it? The acceleration takes place in the line

of action of the resultant force. To find its magnitude use the equation (Art. 14)—

$$R = w\frac{a}{g},$$

where R is 65 pounds' weight and w is $\frac{5}{16}$ pounds' weight; then

$$a = 13 \times 16 \times 32 = 6656^{f}/_{ss}.$$

We often require to find the acceleration which the forces will produce in the particle along a given line, and this can be done by finding the total component which the forces produce along this line, and then using the equation of Newton's Second Axiom—*viz.* $X = w\frac{a_x}{g}$ (Art. 17). Thus, what acceleration along the line $A'A$ will the forces in fig. 22 produce in the particle (5 ounces) at O? The total component, X, which the forces give along this line is 56 pounds' weight, and it acts in the sense OA'; so that, if a_x denotes the acceleration of the particle along OA', we have—

$$56 = \frac{5}{16} \cdot \frac{a_x}{32}$$

$$\therefore \; a_x = \frac{56 \times 16 \times 32}{5}{}^{f}/_{ss}.$$

Similarly, if a_y is the acceleration of the particle along OY,

$$33 = \frac{5}{16} \cdot \frac{a_y}{32}$$

$$\therefore \; a_y = \frac{33 \times 16 \times 32}{5}{}^{f}/_{ss}.$$

We may now lay down the obvious principle that—*the algebraic sum of the components of any number of forces along any line whatever is the same as the component of the resultant of the forces along this line.* For, if the line is OA, and we resolve all the forces along OA and along the perpendicular line OB, the resultant force is actually found by combining into one force the collected component, X, along OA, and the collected component, Y, along OB.

The second method of finding the resultant of any number of forces acting on a particle is as follows :—

Suppose forces P_1, P_2, P_3, P_4, P_5 acting at O, to be represented on any scale by the arrow-marked lines OP_1, OP_2, OP_3, OP_4, OP_5 (fig. 23). Then we get the resultant, Oa, of P_1 and P_2 by taking the diagonal of the parallelogram OP_1aP_2. Now the point a can be obtained without drawing a parallelogram, if we simply draw from P_1 a line, P_1a, equal and parallel to OP_2. The resultant of Oa and OP_3 will be the resultant of the three forces P_1, P_2, P_3; and this is obtained by taking the diagonal, Ob, of the parallelogram determined by the sides Oa and OP_3. Again, we see that the point b can be obtained without drawing the parallelogram, if we simply draw ab equal and parallel to OP_3. Similarly, the resultant of P_1, P_2, P_3, and P_4 is the resultant of Ob and P_4; and this we obtain by drawing bc equal and parallel to OP_4; thus we have Oc as the resultant of P_1, . . . P_4. Finally, the resultant of Oc and P_5 is got by drawing cd equal and parallel to OP_5; and the resultant of the whole set of given forces, P_1, . . . P_5 is completely represented by the line Od.

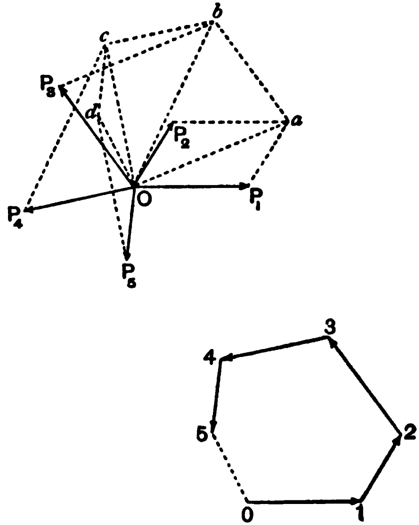


Fig. 23.

It does not matter, of course, *in what order* we take the forces P_1, . . . P_5 in combining them two and two. Thus we might have begun by combining P_2 and P_5 into a single resultant.

It will now be seen that the upper part of fig. 23 contains many more lines than are necessary; and we may proceed, as represented in the lower part of fig. 23, to take any point, o;

from o draw o1 equal and parallel to OP_1; from 1 draw 12 equal and parallel to OP_2; from 2 draw 23 equal and parallel to OP_3; from 3 draw 34 equal and parallel to OP_4; and from 4 draw 45 equal and parallel to OP_5; then—

> *The line* o5 *drawn from the first vertex,* o, *to the last vertex,* 5, *of the figure* o12345, *represents the resultant of the given forces.*

[Of course the resultant acts at O, and is parallel to o5.]

The set of lines o1, 12, 23, 34, and 45 do not form a closed figure; but they form what is called *a polygon of the forces* $P_1, P_2, \ldots P_5$: it is an unclosed polygon, as a rule.

If it were closed, what then? The point 5 would fall on the point o, and the resultant (represented by the line o5) would be zero; that is, the given forces would have *no resultant*; they would, acting together on a particle at O, produce *no acceleration* in it; they would then be said to be *in equilibrium*. Forces which have no resultant are said to be in equilibrium.

We have called the figure o12345 *a* polygon of the given forces, because it is possible to draw several different polygons, all leading to the same result, by taking the forces in various orders.

This is an important thing to understand, so we shall draw a polygon different from that in fig. 23, and show that we still get a closing line exactly equal and parallel to o5 in fig. 23. Thus, suppose that we first draw a line, o3 (fig. 24), equal and parallel to OP_3; then 32, equal and parallel to OP_2; then 25, equal and parallel to OP_5; then 51, equal and parallel to OP_1; and finally 14, equal and parallel to OP_4; we have now the closing line, o4, of the polygon, and this is equal and parallel to o5 in fig. 23.

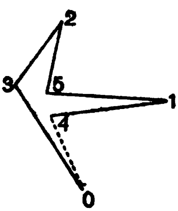


Fig. 24.

It is evident to common-sense that this must be so, because the given forces $P_1 \ldots P_5$ can have only one resultant, since a particle acted upon by them must have a single definite acceleration; and it must be just as allowable to begin by finding a single force to replace P_3 and P_2 as to begin by finding a single force to replace P_1 and P_2. The order in which

we combine the pairs of forces into single forces is whatever we please.

This method of finding the resultant of any number of given forces is called a *graphic method*—the method of the *Polygon of Forces*. Of course, the sides of the force-polygon need not be drawn *equal* to the lines OP_1, OP_2, . . .; they may be all drawn *proportional* to OP_1, OP_2, . . . on any scale.

If the given forces are in equilibrium, any one of them is exactly equal and opposite to the resultant of all the rest. Every one of the force-polygons that can be drawn is, in this case, a closed polygon, since the resultant is always represented in magnitude, direction, and sense by the line

drawn from the first vertex to the last

of any force-polygon; and if the resultant is zero, the last vertex must fall on the first.

A very useful particular case is that of *three* forces in equilibrium, as expressed in the—

20. **Principle of the Triangle of Forces.**—*If three forces are in equilibrium, any triangle whose sides are parallel to their lines of action represents their magnitudes.*

Let OP, OQ, OR (fig. 25) represent three forces acting in

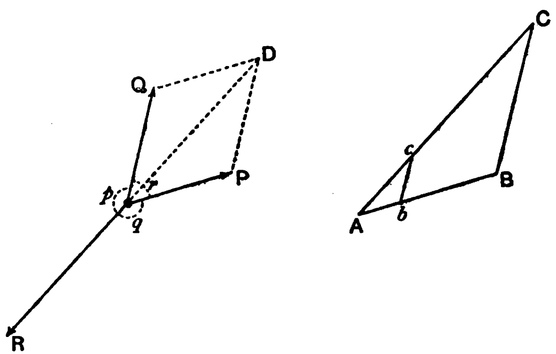


Fig. 25.

equilibrium at O; then, taking any point, A, draw AB parallel to OP and proportional to it on any scale; from B draw BC

parallel and proportional to OQ; and from C draw a line parallel and proportional to OR. The point C must fall on A, since the resultant of OP, OQ, and OR is zero, by what has just been proved.

If bc is drawn parallel to BC, the sides of the triangle Abc are proportional to the forces, so that this triangle is a triangle of forces just as much as the triangle ABC.

Another way of expressing the relations between three forces in equilibrium is this: *when three forces are in equilibrium, each is proportional to the sine of the angle between the other two.*

Let the angle between P and Q in fig. 25 be denoted by r, and the others as represented. Then if we draw the diagonal, OD, of the parallelogram determined by OP and OQ, OR must be equal to OD and a continuation of it. Thus, then, in the triangle OPD we have—

$$P : Q : R = \sin ODP : \sin DOP : \sin OPD.$$

But $\sin ODP = \sin DOQ = \sin p$; also $\sin DOP = \sin q$; and $\sin OPD = \sin (\pi - r) = \sin r$; hence we have—

$$\left.\begin{aligned} &P : Q : R = \sin p : \sin q : \sin r; \\ \text{or,}\quad &\frac{P}{\sin p} = \frac{Q}{\sin q} = \frac{R}{\sin r}. \end{aligned}\right\} \quad . \quad . \quad . \quad (a)$$

This relation we shall sometimes refer to as the *law of sines*, and it is one which is in such frequent use that the student should at once become familiar with it.

There is no such simple relation between four or any greater number of forces in equilibrium; because four forces in equilibrium are represented by the sides of a closed four-sided figure—*i.e.* a quadrilateral; and the sides of a quadrilateral are not proportional to the sines of any angles in the figure.

EXERCISES

1. There are three forces in equilibrium; a particular triangle drawn with sides parallel to the lines of action of the forces is found to have its sides equal to 13, 12, and 5 inches, while the force to which the last side is parallel is known to be 15 pounds' weight; what are the magnitudes of the other two forces?

   Ans. 39 and 36 pounds' weight. (If the two forces in question are denoted by P and Q, we have, by the principle of the triangle of forces, $P : Q \cdot 15 = 13 : 12 : 5$; therefore, etc.)

2. In the last question, what is the angle between the lines of action of the forces 36 and 15?

Ans. A right angle.

3. The sides of a triangle 13, 14, and 15 inches long are parallel to the lines of action of three forces in equilibrium, and the sum of the two forces parallel to the sides 13 and 15 is 140 pounds' weight; find all the forces.

If the forces parallel to the above sides, in order, are P, Q, R, we have—

$$\frac{P}{13}=\frac{Q}{14}=\frac{R}{15}, \text{ and also } P+R=140.$$

Now, if we put $P=13x$, $Q=14x$, $R=15x$, where x is any quantity, we have $P+R=28x=140$, $\therefore$ $x=5$; so that $P=65$, $Q=70$, $R=75$ pounds' weight.

4. Show that if a particle is kept at rest by three equal forces the angles between their lines of action are each equal to 120°.

5. Three forces, whose magnitudes are P, $P+Q$ and $P-Q$, act on a particle, and their lines of action include three angles each of 120°; show that the magnitude of the resultant is $Q\sqrt{3}$, and that the resultant is at right angles to P.

6. Three forces, whose magnitudes are proportional to m^2+n^2, $2mn$, and m^2-n^2, m and n being any magnitudes, keep a particle at rest; show that two of them are at right angles to each other.

7. Given the direction of the resultant of two forces, and the magnitude and direction of one of them, find the least magnitude of the other and the corresponding direction.

(The corresponding direction is at right angles to the resultant.)

21. **Conditions of Equilibrium.**—When any number of forces act together on a body and have no resultant, they are said (as has been already stated) to be *in equilibrium*. There are many ways in which we may express the conditions of equilibrium of a body acted upon by forces, but they all come to the same thing—they are merely different ways of saying that *the forces have no resultant*. For the present we confine our attention to a particle, so that the forces with which we are dealing may be considered as all acting at one point. For example, if a particle is acted upon by only *two* forces, what is the condition of equilibrium? *The two forces must be equal and opposite in the same right line*, because this is the only way in which they can produce a zero resultant.

If a particle is acted upon by *three* forces, what are the conditions of its equilibrium? If any triangle is drawn with its sides parallel to their lines of action, the forces must be proportional to these sides, respectively, and their senses must be same as those of the sides taken continuously round the triangle. Thus, if the forces acting at a point, O (fig. 26), are represented on any scale by the sides AB, BC, CA of a triangle ABC (fig. 27), these sides being taken in what is called *cyclical order*, the forces at O are in equilibrium. On the other hand, if the forces at O act in the senses represented by the arrows in fig. 28, although the magnitudes are proportional to the sides AB, BC, CA, they are *not* in equilibrium; and observe that the sense of the force parallel to BC is opposed to the cyclical order of the senses of the other two. The forces at O, if in the senses of the arrows in fig. 28, would have a resultant represented by $2.CB$.

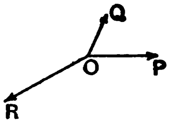


Fig. 26.

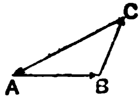


Fig. 27.

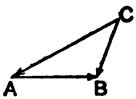


Fig. 28.

Three forces in equilibrium satisfy, of course, the law of Sines given in Art. 20.

For any number of forces, the resultant will completely vanish if *the sum of the components of all the forces along every line is zero*, because we know from Art. 19 that we obtain the resultant by collecting the total component, X, of the forces along *any* one line, and the total component, Y, along the perpendicular line; and then—

$$R^2 = X^2 + Y^2 \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

where R is the resultant; so that R will be zero if

$$X = 0 \text{ and } Y = 0, \quad . \quad . \quad . \quad . \quad (\beta)$$

—*i.e.* if the sum of the components of the forces is zero along any line and also along its perpendicular line. It is obvious that *the resultant will be zero if the sum of the components of the forces is zero along each of two different directions, even though these directions are not at right angles.* Conversely, *if the resultant is zero, the forces have no total component along any line whatever*; for, if they have a total component, X,

along any line, equation (a) shows that R cannot be zero, no matter what component, Y, they may have along the perpendicular line.

22. **Property of Moments.**—Not only has the resultant of two forces the same component along any line as that given by the two forces themselves (see p. 32), but it has another property in common with the two forces.

Let OP (fig. 29), represent completely a force P, and let A be any point, and p the length of the perpendicular from A on OP. If we imagine the force P as acting on a body —say a thin rigid membrane lying in the plane of the paper—and suppose that A is a point in this body so that an axis can be stuck through the body at A perpendicularly to the plane AOP, this axis being held fixed, what will be the effect produced on the body by the action of P? Obviously a *rotation*, or turning, round the axis; and it is evident that the rapidity of this rotation will depend on two distinct things—

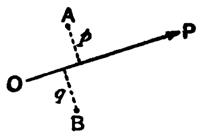


Fig. 29.

(a) the magnitude of the force P;
(b) the length of the perpendicular p.

The student will see, long after he has begun this subject, that the rotation will depend on the product—

$$P \times p. \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Our object now is not to show how this product is connected with rotation, but to prove that it has certain simple properties.

Observe, however, that the equilibrium and motion of a particle can be completely discussed without any reference to this notion of rotation—which belongs specially to the nature of bodies of large dimensions.

Definition.—*The "moment" of a force about a point is the product of the force magnitude and the perpendicular from the point on the line of action of the force.*

Then if in fig. 29 $P=20$ pounds' weight and $p=8$ inches, the moment of P about A is 160 inch-pounds' weight.

The moment of a force about a point has not only ***magnitude***, but ***sign***; and the sign is determined by the ***sense of the rotation*** which the force would produce round the point. To determine the sense of the rotation, hold the point of a pencil at the point, the pencil being perpendicular to the plane of force and point; imagine that this pencil is an axis stuck through a body on which the force acts, and see whether the body would rotate in the sense of watch-hand rotation or in the reverse. Thus, in the above figure the rotation would be opposite to that of the hands of a watch whose face is towards us. These rotations are designated as "clockwise" and "counterclockwise"; so that the moment of P about A is 160 inch-pounds' weight counterclockwise. We shall frequently, in the sequel, indicate this in the following way:

$$160$$

Take the point B in fig. 29. If the perpendicular from B on P is 10 inches, the moment of P about B is

$$200$$

the bent arrow indicating a clockwise sense.

The moment of a force about any point on its own line of action is zero.

Varignon's theorem of moments. The sum (with their proper signs) of the moments of two forces about any point in their plane is equal to the moment of their resultant about the point.

Observe that the moment $P \times p$ of the force represented by OP (fig. 29) about A is graphically represented by double the area of the triangle AOP which has the point A for vertex and the force for base.

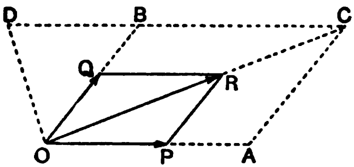


Fig. 30.

Let OP and OQ (fig. 30) represent any two forces, P and Q, and OR their resultant, R, according to the parallelogram law; let D be any point, and let it be required to show that the moment of R about D is equal to the sum of the moments of P and Q about D. From D draw DC parallel to one of the forces, P; produce OR to meet

DC in C, and OQ to meet DC in B, and draw CA parallel to BQ, meeting OP in A. Then, obviously, $OA : AC : OC = OP : PR : OR$, so that we may adopt a new scale for representing the forces, and take OA, OB, OC to represent P, Q, R, respectively.

Then the moment of OC about D is represented by $2.DOC$, where DOC means the area of the triangle DOC; moment of OB about $D = 2.DOB$; moment of OA about $D = 2.DOA = 2.COA$ (since DOA and COA are on the same base and between the same parallels) $= 2.BOC$; hence sum of moments of OA and OB about $D = 2.DOB + 2.BOC$, which $= 2.DOC$, and this is the moment of OC about D; therefore, etc.

If D is between the lines OP and OQ, the moments of P and Q about it are of opposite signs, and the arithmetical *difference* of their moments about D is equal to the moment of R about it.

This is the second property which is common to any number of forces and their resultant, the first being that given in p. 32: *The sum of the moments of any number of forces acting in one plane about any point in the plane is equal to the moment of their resultant about that point;* for, we may replace two by their resultant, and take the moment of this force and another of the given forces; and so on.

If we had defined the moment of a force about a point as the product $P \times p^2$, or as anything else than $P \times p$, this theorem concerning forces and their resultant would not be true.

If the sum of the moments (with proper signs, clockwise being $+$, suppose, and counterclockwise being $-$) of any number of forces about a given point is zero, what do we know about that point? That it must lie on the resultant of the forces.

EXERCISES

1. ABC is a triangle whose sides, AB, BC, CA, are 14, 13, and 15 inches; a force of 15 pounds' weight acts along the side BA in the sense BA; what is the moment about C?

Ans. 180 inch-pounds' weight. (Indicate the sense of the moment.)

2. A and B are two given points 20 inches apart; a force passing through A has a moment of 300 inch-pounds' weight about B; what must be the

direction of this force so that its magnitude may be least, and what is the least magnitude?

Ans. The force must act at right angles to AB, and then its magnitude is 15 pounds' weight. To see this, let the force have any direction; let p be the length of the perpendicular from B on its line of action, and let P be the magnitude of the force. Then $B \times p = 300$,

$$\therefore \; P = \frac{300}{p},$$

so that P will be least when p is greatest. But the greatest perpendicular from B on a line passing through A is BA; therefore the force when it is least acts at right angles to BA. Putting $p = BA = 20$, we have $P = \frac{300}{20} = 15$.

Generally, if the force passing through A has a given moment, M, about B, the least value of the force is $\dfrac{M}{AB}$.

3. In the last represent graphically the magnitudes of all the forces passing through A which have the given moment 300 inch-pounds' weight about B.

At A draw a perpendicular to AB, and measure off on this line a length AO to represent $\frac{300}{20}$, or 15, pounds' weight; from O draw a line OL parallel to AB; then all the required forces are represented on the same scale by lines drawn from A to points on OL. This follows from the graphic representation of a moment by the area of a triangle. (See p. 40.)

4. Given the moments, M and N, of a force about two given points, A and B, show that one point on the line of action of the force is known.

The force must pass through the point O on AB such that $\dfrac{AO}{OB} = \dfrac{M}{N}$.

5. Show that if the moments of a force about any three points, A, B, C, not in a right line, are given, the magnitude and line of action of the force are known.

6. ABC is a triangle such that $AB = 14$ inches, $BC = 13$, $CA = 15$; if M_A, M_B, M_C denote the moments of a force, P, about A, B, C, and if

$$M_A = 216 \text{ inch-pounds' weight}$$

$$M_B = 120 \quad ,, \quad ,, \quad ,,$$

$$M_C = 120 \quad ,, \quad ,, \quad ,,$$

show that P is 26 pounds' weight, and that its line of action is the parallel to BC drawn through the foot of the perpendicular from C to AB.

EXAMINATION ON CHAPTER III

1. What is meant by the *resultant* of two forces? How is it found?

2. If the resultant of two forces, P and Q, has the magnitude $P+Q$, what can be inferred? If the resultant has the magnitude $P\sim Q$, what can be inferred?

3. Can two forces whose magnitudes are 20 and 16 act at such an angle that their resultant is 40? Can they act so that their resultant is 3?

4. What are meant by *components* of a given force?

5. Can a given force be resolved into two components, each of which is equal in magnitude to the force itself? [This question is the same as: Can a given line be made the base of a triangle, each side of which is equal to the given line?] Exhibit the lines of action, and say what the angles between them are.

6. What is meant by the *rectangular component* of a given force along a given line? How is the *sense* of this component determined?

7. How is the resultant of any number of *concurrent* (*i.e.* passing through the same point) forces found by replacing each by two components?

8. When any number of forces act on a particle of given mass, how do you find the direction and magnitude of the resultant acceleration of the particle?

9. How do you find the magnitude and sense of the acceleration of the particle along any given line?

10. How is the resultant of any number of concurrent forces found by the graphic method of the *polygon of forces*?

11. When several forces act together at a point, why do we speak of *a* polygon of the forces rather than of *the* polygon of the forces?

12. If the forces have no resultant, what happens to each polygon of the forces?

13. Can two forces be in equilibrium?

14. State in three different ways the conditions of equilibrium of *three* forces.

15. State concisely the condition of equilibrium of any number of forces acting on a particle. [No component along any line.]

16. Into how many components can a given force be resolved? [An infinite number; because a polygon of any number of sides can be described on a line representing the force. "Components" here, of course, do not mean *rectangular components*.]

17. What is meant by the *sense* of the moment of a force about a point?

18. How is the magnitude of the moment of a given force about a point graphically represented?

19. If a force passing through a given point has a constant moment about another given point, when is the magnitude of the force least?

20. Represent graphically the magnitudes of all forces passing through a given point when the forces have the same moment about another given point.

CHAPTER IV

PARTICLES ON SMOOTH PLANES (NEWTON'S THIRD AXIOM, OR LAW, OF MOTION)

23. **Definition of Smooth Surfaces.**—Suppose that a body, M (fig. 31), which is acted upon by any forces, rests at a point P against a surface AB. Then it is obvious that this surface is exerting (or may be exerting) a force on the body at the point P, because, in general, the state of equilibrium or motion of M will not be the same if the surface AB is removed as when AB is not removed: the surface AB is preventing the body M from going through it.

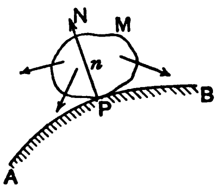


Fig. 31.

Now, if the surfaces in contact at P—*i.e.* the surface of M and the surface AB—are such that the only force which AB can exert on M is one confined to the direction of the normal Pn to the surface of contact, the surfaces are said to be *smooth*. This gives us a definition of smooth surfaces—*viz.*:

> *Two surfaces are said to be smooth if the force which either of them exerts on the other, when they are placed in contact, can act in no other direction than that of the normal to their surface of contact.*

Experience makes us acquainted with surfaces which nearly satisfy this condition. Thus, if a glass rod, GP (fig. 32), with a rounded end is pressed along its length GP against a well-cleaned glass plate, AB, at the point P, it will be found impossible to keep the rod from slipping in any position (such as that represented in the figure) in which it is inclined to the normal, Pn, to AB at P. In other words, the glass

plate can exert no force along AB to prevent the slipping of the rod—although it can exert a good deal to prevent penetration. Many other instances of the same kind will readily present themselves.

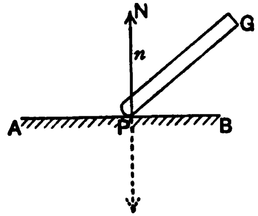


Fig. 32.

But, of course, surfaces such as we have just defined exist *strictly* only in imagination: no surfaces can be found which will not offer *some* (however slight) resistances along their tangent planes. We meet in experience with surfaces which are *very nearly*, but not quite perfectly, smooth; and such surfaces give us the notion, or suggestion, of perfect smoothness.

We shall seriously assume the existence of such surfaces for the present.

Once we give the surface AB credit for exerting the force, represented by N, on the body, we may remove AB out of the figure and use the force N instead of it. This is an unusual thing to do; but nevertheless, once the force N is introduced into our figure, it is well to remember, there is no need for the surface which produced it.

24. **Newton's Third Axiom, or Law, of Motion.**—Now it is quite obvious that in the case of fig. 31, not only is the surface AB exerting a force on the body M, but also the body M is exerting a force on the surface AB. Newton assumes that this latter force is *exactly equal and opposite to the former*. Thus, if the normal nP is produced through P, the force which M exerts on AB is equal to N in magnitude, and it acts in the reverse sense, nP. Similarly, in fig. 32, the table AB experiences a force exactly equal to N but in the downward direction.

Whenever any two bodies, A and B, are in contact, and B exerts a force F on A, then A exerts the same force F on B, but in exactly the reverse direction. If we denote the first by F, we may call the second $-F$. This is an axiom which we assume with Newton; but Newton goes much farther (or *apparently* much farther) in his assumption. He says, in fact, that *it is impossible for any force to exist in the Universe*

without the existence, at the same time, of another force equal and opposite to it in the same right line—just as impossible, in fact, as a straight line without two ends to it.

We may have no difficulty in admitting this in the case of actual contact between two bodies—as when the finger presses the table, or as when a heavy body rests on the ground: the ground exerts an upward push on the body, and the body a downward push on the ground. But the axiom holds good for cases in which bodies are not *in what we commonly call actual contact.* Thus, the Sun exerts on the Earth, at a distance of about 92 millions of miles, a force which we can easily calculate in any units we please—say tons' weight. The calculation gives us the result that the Sun is perpetually pulling at the Earth with a force of nearly—

$3\frac{1}{2}$ millions of billions of tons' weight.

Newton assumes that the Earth is perpetually pulling at the Sun with exactly the same force in the same right line (that joining their centres): the two forces are equal and opposite, just as in the case of the finger acting on the table and the table reacting on the finger. Moreover, there is probably no essential difference between the "contact" of the finger with the table and the "contact" of the Earth with the Sun: there is probably no contact at all in either case, but merely the intervention of a medium—of much less thickness, truly, between the finger and the table than between the Earth and the Sun—but still an intervening medium which enables the action to be exerted in both cases. Newton's Third Axiom, or Law, of Motion, as enunciated in the "Principia," is this:

> *To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal and directed to contrary parts.*

This Axiom is commonly summed up in the words: "Action and reaction are equal and opposite," which are obviously far from expressing all that is contained in Newton's own words.

If a body, B, exerts a force, F, on a body, A (either by what we call *contact*, or by any other means), the body, A,

exerts on B the force $-F$, a force equal and opposite to F *in the same right line.*

It is important to note that these forces—called "action" and "reaction"—act in the same right line. Thus, in fig. 33, if the body B exerts on A a force represented by the arrow m, the body A cannot exert on B a force represented by the arrow n, although these two forces are equal, parallel, and of opposite senses; they must be equal and opposite *in the same right line*,* as in fig. 34.

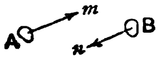


Fig. 33.

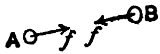


Fig. 34.

Newton's Third Axiom asserts, then, that such a thing as an isolated force does not exist in the universe: whenever any body, A, is experiencing a force F, some other body, B, is experiencing the force $-F$, equal to the previous force and opposed to it in the same right line.

25. **Tension of a Cord.**—The nature of action and reaction, as consisting of equal and opposite forces, is exemplified in the case of a tight cord. As a very simple example, suppose that we have a cord, AB, fig. 35, one end, A, of which is attached to a fixed peg, while from the other end, B, is suspended a mass of weight W. Now, suppose that we take any point, P, in the cord, and consider the action exerted on the portion AP of the cord by the lower portion. This action consists of a *downward* pull, represented by the force T in the middle part of fig. 35. If, on the other hand, we consider the action exerted on the lower portion by the upper, we see that it consists of an *upward* pull, represented by the force T at the right of fig. 35. This is exactly the same force as that exerted upon the portion AP, but reversed in direction; and we see in fig. 35 that T must be equal to W if the weight of the portion PB of the cord is negligible: if it is not negligible, T is equal to $W+$the weight of the portion PB of the cord.

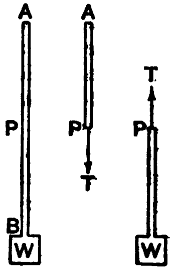


Fig. 35.

* The neglect of this consideration has led to erroneous substitutes for Newton's Laws—as we may have occasion to point out farther on.

This force T is called the *tension* of the cord at P. Its magnitude will be equal to W at every point of the cord if the weight of the cord is negligible.

What, then, is the sense in which the tension acts at any point, P, of the cord?

The answer depends on whether we are considering the equilibrium of the upper portion, AP, or that of the lower portion of the cord. If the former, the tension is a *downward* force; if the latter, it is an *upward* force. The case is just the same as that of the finger pressing the table. What is the direction of the force exerted between the finger and the table? As acting on the finger, it is an upward force; as acting on the table, it is a downward one. In the case of a cord kept tight, either as in fig. 35 or in any other way, we may suppose the tension at any point, P, indicated by a spring balance, if we sever the cord at P and attach the ends of the spring to the upper and lower parts of the divided cord: the hand, or movable index, of the balance will then point out the magnitude of the tension of the cord at P.

The beginner must not fall into the error of saying that, because the tension T, at P, is a downward force, and is also an equal upward force, it can be ignored, since two equal and opposite forces applied at the same point cancel each other. Observe that these two equal and opposite forces *do not act on the same body*: one of them acts on the upper portion, AP, of the cord, while the other acts on the lower portion.

The force which the cord exerts on the peg A is a downward one (equal to W, if the weight of the cord is negligible), while the peg exerts on the cord an equal upward force. The wall into which the peg is fixed exerts an upward force on the peg to balance or *equilibrate*, as it is said, the tension acting on the peg.

Thus, then, in all cases, ***whether we are to represent the tension of a cord at any point as a force acting in one sense or in the opposite depends on the body, or portion of the cord, whose equilibrium or motion we are considering.***

The beginner should pay great attention to this.

The case in which a cord passes over or under any number of *smooth* pegs, or other surfaces, is one which we must notice. Suppose that a cord, the weight of any portion or the whole of which is negligible, has one end fixed at a point, A (fig. 36),

and that, passing round any number of fixed smooth pegs or axes, *B*, *C*, *D*, the other end, *B*, is pulled with any force, *P*; then the various parts, *AB*, *BC*, *CD*, *DE*, of the cord may be regarded as straight lines (which, if the cord has weight, they cannot be); and, the pegs being quite smooth, the magnitude of the tension is the same at every point, *Q*, *R*, *S*, . . . of the cord, and is, of course, equal to the tension, *P*, at *E*. That this tension is constant we will assume; much later in the subject the student will see why this is so. We assume, then, that when the surfaces over which a cord passes are smooth, and the effect of gravity on the cord is negligible, if a small spring balance were inserted between the severed portions of the cord at any point, *Q*, *R*, *S*, . . . , this balance would indicate the *same tension* at all such points.

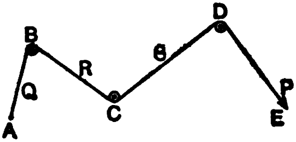


Fig. 36.

This constancy of the tension holds only in one and the same *continuous* cord. If the cord is *knotted* at any point to another cord, the tensions in the two parts of the cord which start from the knot are, in general, different. Thus, in fig. 37, if the cord, *ABCD*, is knotted at a point, *C*, to another cord, *CF*, and if from the portion hanging over the peg, *D*, a body of weight *W* is suspended, the tension will be equal to *W* at every point in the portion *CD*; but there will be a different tension, *S*, in the portion *CB*; and, again, a different tension, *T*, in the cord *CF*. The tensions *T*, *S*, *W* are represented in the figure as they act on the knot *C*. The tension is *S* at all points in *AB*.

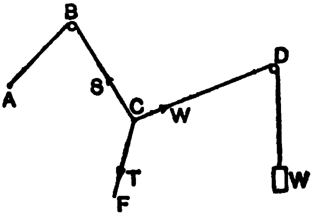


Fig. 37.

The magnitudes of *S* and *T* can be found in terms of *W*, by considering the equilibrium of the knot *C*, when we are given the angles between the cords at *C*, by the *law of Sines* or the *triangle of forces* (Art. 20).

What would result in fig. 37 if the knot *C* were replaced by a small smooth ring? The tension, *S*, would then become

equal to W, since the cord $ABCD$ would then be *continuous*, and would at C merely pass under a smooth surface—*viz.* the inner surface of the ring.

26. **Pressure on a Peg.**—As further illustrating the nature of the tension of a cord, take the case represented in fig. 38.

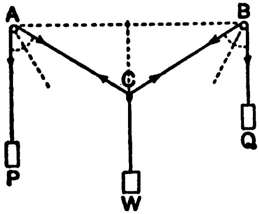


Fig. 38.

Let A and B be two smooth pegs fixed in a horizontal line; let a cord, with masses P and Q attached to its ends, pass over these pegs; and let a small ring, C, to which a mass, W, is attached, be threaded on the cord between A and B. It is required to find the position of equilibrium and the pressures on the pegs.

Now, firstly, observe that the cord is continuous and passes over and under smooth surfaces, so that *its tension must be everywhere the same.* At one extremity the tension is obviously the weight of P, and at the other it is the weight of Q; therefore no equilibrium is possible unless

$$P = Q.$$

Consider now the equilibrium of the ring. It is maintained by three forces—*viz.* W, and the two equal tensions (each equal to P) represented by the arrows drawn from C. Hence the line of action of W must bisect the angle C (Cor., Art. 16); therefore C will settle down on the vertical line which bisects AB, and the angles CAB and CBA are equal. Denote each of them by θ. Then the position of equilibrium will be known if θ is known.

Now, resolving vertically for the equilibrium of the ring we have—

$$P \sin\theta + P \sin\theta - W = 0$$

$$\therefore \sin\theta = \frac{W}{2P}. \qquad . \qquad . \qquad (1)$$

This determines the position of equilibrium.

Now what forces keep the peg A in equilibrium? The tensions of the two portions of the cord leaving A, which are represented by the arrows drawn along these lines from A, and

the force supplied by the wall in which the peg is fixed. This latter is equal and opposite to the resultant of the two tensions, and this resultant (Cor., Art. 16) bisects the angle between the lines AP and AC. If R is the resultant of the two forces P, P which include the angle $\frac{\pi}{2} - \theta$, we have (Art. 16)—

$$R^2 = P^2 + P^2 + 2P^2\cos\left(\frac{\pi}{2} - \theta\right)$$
$$= 2P^2(1 + \sin\theta)$$
$$= 2P^2\left(1 + \frac{W}{2P}\right), \text{ by (1)};$$
$$\therefore\ R = \sqrt{P(2P + W)}. \quad . \quad . \quad . \quad . \quad (2)$$

This is the resultant pressure exerted on the peg by the cord, and it is counteracted by the wall. Observe that there is no position of equilibrium unless $W < 2P$; for (1) requires this condition in order that θ should be a real angle. The reason of this is easily seen; for if W were equal to $2P$, the ring would be obliged to fall down an infinite distance to enable the two parts of the cord, CA and CB, to close up into one right line.

As numerical illustrations the following may be taken:—

1. AB is 24 inches, $W = 20$ pounds, P (and therefore Q) $= 26$ pounds.
   Then, in the position of equilibrium C is 5 inches below the middle point of AB, and the resultant pressure produced by the cord on each peg is $12\sqrt{13}$, or 43.27, pounds' weight.
2. AB is 16 inches, $W = 60$ pounds, $P = Q = 34$ pounds.
   In the position of equilibrium C is 15 inches below the middle point of AB, and the resultant pressure on each peg $= 16\sqrt{17} = 65.92$ pounds' weight.

27. **Principle of Separate Equilibrium or Motion.**—Whenever a system, at rest or in motion, consists of several parts, we may consider the equilibrium or motion of any part of the system as if the rest of the system did not exist, *provided that we produce on the part considered all the forces which are actually produced on it by the other parts which we have imagined to be removed.*

This has already been illustrated in Art. 23, in which we considered the surfaces AB in figs. 31 and 32 as removed, and the forces which they exert on the bodies in contact with them supplied by some other means.

This principle is an axiom which runs through the whole of mathematical physics.

28. **Body on Smooth Inclined Plane.**—Let a body of weight W (fig. 39), be placed on a smooth plane inclined to the horizon at the angle i, and suppose that a cord is attached to the body, and pulled with a tension P in a direction making the angle θ with the upper surface of the plane. What is the value of P which will keep the body in equilibrium? Let N be the reaction of the plane on the body; then N acts along the normal to the plane; and the body is in equilibrium under the action of only three forces. Hence, as P must be expressed in terms of the given force W, we use the law of Sines (Art. 20) thus:

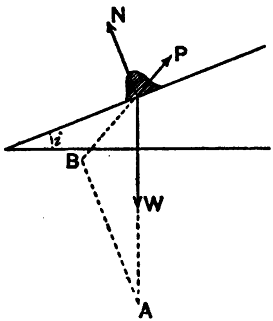


Fig. 39.

$$\frac{P}{W}=\frac{\sin W\hat{N}}{\sin N\hat{P}}$$

where $W\hat{N}$ means the angle between N and W. But

$$W\hat{N}=\frac{\pi}{2}+\frac{\pi}{2}-i=\pi-i;\text{ and } N\hat{P}=\frac{\pi}{2}-\theta;\text{ hence}$$

$$P=W\frac{\sin i}{\cos \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the same way—

$$\frac{N}{W}=\frac{\sin W\hat{P}}{\sin P\hat{N}}=\frac{\sin\left(\frac{\pi}{2}+i+\theta\right)}{\cos \theta}=\frac{\cos(i+\theta)}{\cos \theta}. \quad . \quad (2)$$

Of course, these could be obtained by resolving along and perpendicular to the plane for the equilibrium of the body.

The values of N and P can be graphically represented in the figure by means of the *triangle of forces* (Art. 20). Any triangle with its sides parallel to the forces will represent their relative magnitudes; hence draw any line, AB, parallel to N, meeting the lines of action of W and P in A and B; then, if O is the point at which P, N, and W meet,

$$P:N:W = OB:BA:AO.$$

As a numerical example, let $W=104$ pounds, $i=\tan^{-1}\frac{5}{12}$, $\theta=\tan^{-1}\frac{3}{4}$.

Then $P=104\dfrac{\frac{5}{13}}{\frac{4}{5}}=50$ pounds' weight; $N=104\dfrac{\frac{12}{13}\cdot\frac{4}{5}-\frac{5}{13}\cdot\frac{3}{5}}{\frac{4}{5}}=$ 66 pounds' weight.

Supposing that in this numerical case the cord is pulled, not with the tension 50 pounds' weight which corresponds to equilibrium, but with a force of 20 pounds' weight, what will happen?

Resolve the forces along and perpendicular to the plane; then, acting down the plane we have a component, X, given by the equation—

$$\begin{aligned} X &= W\sin i - P\cos\theta = 104\times\tfrac{5}{13} - 20\times\tfrac{4}{5} \\ &= 24 \text{ pounds' weight;} \end{aligned}$$

and acting in the normal in the sense of N the total component, Y, is given by—

$$\begin{aligned} Y &= N + P\sin\theta - W\cos i \\ &= N + 12 - 96 \\ &= N - 84. \end{aligned}$$

Now, the reaction of the plane can have any value, except a negative one; and to prevent motion of the body in the direction perpendicular to the plane, the plane has merely to react with a force N equal to 84 pounds' weight, which it will do; but there is an unbalanced component force of 24 acting on the body down the plane. Hence the body will slide down with uniformly accelerated motion, in which

the acceleration a f/ss is due to the action of a force 24 on a body of weight 104, so that (Art. 17)—

$$a = \frac{24}{104} g = \frac{96}{13} = 7\tfrac{5}{13} \text{ feet per sec. per sec.}$$

If in the general case, represented in fig. 39, $P=0$—*i.e.* if a body is placed on a smooth inclined plane, and not acted upon by any force except its own weight and the reaction of the plane—the body will remain on the plane, but slide down with uniformly accelerated motion.

For, if W is resolved along and perpendicular to the plane, the forces in these directions acting on the body are—

$$W \sin i \text{ and } N - W \cos i.$$

The plane will react with a force N equal to $W \cos i$ to preserve equilibrium along the normal, but the down-plane force, $W \sin i$, is unbalanced, and the acceleration, a, which it produces in the body is given by—

$$\begin{aligned} a &= \frac{W \sin i}{W} g \\ &= g \sin i. \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

EXERCISES

1. A and B (fig. 40) are two smooth pegs fixed in a horizontal line; a cord, ACB, passes over them and is attached at its ends to two masses of weights P and Q, while at a point C this cord is knotted to another, from the free end of which a mass of weight W is suspended; find the position of equilibrium and the resultant pressure on each peg.

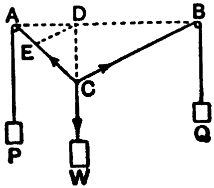


Fig. 40.

The tension at all points in the portion CA is constant, and therefore $=P$; the tension at all points in $CB = Q$.

Consider the equilibrium of the knot C. It is maintained by three forces, as represented by the arrows. Hence we may use the principle of the *triangle of forces*: any triangle with its sides parallel to the three forces acting on C has its sides proportional to their magnitudes. Let the vertical line through C meet AB in D; from D draw $DE \parallel CB$; then the sides of the triangle CDE are proportional to W, Q, and P; that is $CD : DE : EC = W : Q : P$; hence all the angles of this triangle are known by trigonometry. For example, $\cos ECD = \dfrac{W^2 + P^2 - Q^2}{2PW}$;

but $\cos ECD = \sin BAC$, $\therefore$ the angle BAC is known—*i.e.* the direction of the cord AC is known. Similarly, $\cos EDC = \frac{W^2 + Q^2 - P^2}{2\,QW}$; but $\cos EDC = \sin ADE = \sin ABC$, $\therefore$ the direction of the cord BC is known; and since the directions of AC and BC are known, the point C is known, and the position of equilibrium is $\therefore$ known. Does it matter how long the portions AP and BQ are made after the positions of AC and BC are known? No; because we have neglected the weight of every portion of the cord, and therefore, whether P is suspended one inch or one foot below A, the state of affairs will be the same. As a numerical illustration, let $W = 63$ pounds, $P = 52$, $Q = 25$.

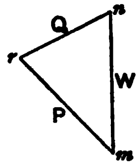


Fig. 41.

Construct a triangle, *mnr* (fig. 41), with the side *mn* vertical, and and let its sides *mn*, *nr*, *rm* be made equal (or proportional) to 63, 25, and 52 units. Then, by trigonometry—

$$\cos rmn = \frac{63^2 + 52^2 - 25^2}{2 \times 63 \times 52} = \frac{12}{13}; \quad \cos mnr = \frac{63^2 + 25^2 - 52^2}{2 \times 63 \times 25} = \frac{3}{5}.$$

The sides of this triangle, *mnr*, are parallel to the sides of the triangle CDE which we wish to find; *i.e.* the cord AC is parallel to *rm*, and BC is parallel to *nr*.

Hence—

$$\sin BAC = \frac{12}{13}, \ \sin ABC = \frac{3}{5}.$$

Now the peg A is acted upon by two forces each equal to 52 pounds' weight—*viz.* the tension of AP and AC—and their resultant bisects the angle PAC. If their resultant $= R$, we have (Art. 16)—

$$R^2 = 52^2 + 52^2 + 2 \times 52^2 \cos PAC = 2 \times 52^2 \left(1 + \frac{12}{13}\right)$$
$$\therefore R = 260\sqrt{\tfrac{2}{13}} = 20\sqrt{26} = 101.98 \text{ pounds' weight.}$$

Similarly, if S is the resultant pressure on the peg B,

$$S = 20\sqrt{5} = 44.72 \text{ pounds' weight.}$$

2. If the knot C is replaced by a small ring from which W is suspended, how is the question of equilibrium affected?

In this case the cord ACB is continuous, and its tension (supposing its weight still negligible) must be the same throughout. Hence no equilibrium will be possible unless P and Q are made equal. In this case the line of action of W will bisect the angle ACB between the two equal tensions at C—*i.e.* D is the middle point of AB; and then, by considering the equilibrium of C, and resolving vertically, we have—

$$2P\cos ACD - W = 0$$
$$\therefore \sin BAC = \sin ABC = \frac{W}{2P},$$

which determines the position of equilibrium.

3. If a body of weight W is placed on a smooth inclined plane of inclination i and is kept at rest by a cord attached to it, the cord being parallel to the plane, find the tension of this cord and the pressure on the plane.

If T = tension of cord and N = reaction of plane, we have—

$$T = W \sin i,$$
$$N = W \cos i.$$

4. If a body of weight W is placed on a smooth inclined plane of inclination i, and is kept at rest by a cord attached to it, the cord being horizontal, find the tension and the pressure on the plane.

Result. $T = W \tan i,$
$N = W \sec i.$

5. If a body whose mass is 12 pounds is placed on a smooth inclined plane the tangent of whose inclination to the horizon is $\frac{5}{12}$, and has attached to it a cord parallel to the plane, the cord being pulled in the upward direction, find the acceleration of the body along the plane when the tension of the cord is—

(*a*) 39 pounds' weight,
(*b*) 26 ,,

Result. In case (*a*) the acceleration is upwards and $= 8\ ^f/_{ss}$;
,, (*b*) ,, downwards ,, $= 5\frac{1}{3}\ ^f/_{ss}$.

The pressure on the plane is the same in both cases, and is $\frac{144}{13}$ pounds' weight.

(Note that, following the work of the example in Art. 19, there is in case (*a*) a total upward component of force along the plane equal to 3 pounds' weight, and in case (*b*) a total downward component of 2 pounds' weight.)

6. If on the same incline there is placed a mass of 52 pounds, and a horizontal cord is attached to it and pulled with a force of 26 pounds' weight, find the acceleration produced in the body.

Result. An acceleration upwards along the plane of $\frac{32}{13}\ ^f/_{ss}$. (We suppose the cord to be pulled towards the right in fig. 39. If it were pulled towards the left the acceleration would be in the downward direction along the plane and 11 times the above value.)

29. **Homogeneity of Expressions.**—Dynamical problems usually involve quantities of different kinds—for example, *forces*, *lengths*, *angles*. The ratio of any two quantities of the same kind is a mere *number*. Now, let us suppose that two

forces are P and W, two lengths a and b, and that θ is the circular measure of an angle; then such an expression as—

$$P = W^2 \frac{a+1}{b^3} \sin\theta \quad . \quad . \quad . \quad (1)$$

is utterly erroneous for the following reasons:

(1.) Since P is a force and W a force, P must be equal to W multiplied by some pure number. P must not be represented as equal to a numerical multiple of W^2, W^3, . . . or any power of W other than the first power.

(2.) If a is a length, which may be measured in *any* units (inches, feet, miles, etc.), we cannot add 1 (*i.e.* the number 1) to a, so that the expression $a+1$ is erroneous.

(3.) Even if $a+1$ were admissible, the expression $\frac{a+1}{b^3}$ is not a *number*, since a is a length and b^3 is the cube of a length, *i.e.* a *volume*; and hence P could not be equal to W multiplied by such an expression as $\frac{a+1}{b^3}$.

There is no objection to the term $\sin\theta$ as a factor multiplying W to express the value of P.

The two sides of equation (1) are not *homogeneous*, *i.e.* of the same kind.

A correct expression would be—

$$P = W \frac{a}{b} \sin\theta, \quad . \quad . \quad . \quad (2)$$

and another correct expression would be—

$$P = W \frac{a^3}{b^3} \sin\theta, \quad . \quad . \quad . \quad (3)$$

since $\frac{a^3}{b^3}$ is the ratio of quantities of the same kind (cubes of length).

Supposing that Q is another force magnitude, a correct expression would be—

$$P = \frac{W^3}{Q^2} \cdot \frac{a^2}{b^2} \sin^3 \theta, \quad . \quad . \quad . \quad (4)$$

since $\frac{W^3}{Q^2}$ is $\frac{(\text{force})^3}{(\text{force})^2}$, *i.e.* force, and therefore the same thing in kind as P; moreover, $\frac{a^2}{b^2}$ is a pure number, and so is $\sin^3\theta$.

There are few things in Dynamics more important for the beginner than an attention to the *homogeneity* of equations, because an eye that is accustomed to look for homogeneity in every expression will at once detect any error that may be contained in it, and thus, with almost no trouble whatever, locate the error in the work which has led to such an expression. The ability to detect errors by the *test of homogeneity* in physical calculations is an object the attainment of which will repay all the labour that may be expended on it.

30. **Things that are given and things to be found in a problem.**—Let the student, in considering any problem, distinguish carefully at the outset two classes of magnitudes which the problem involves—*viz.*:

1. Magnitudes that are given (called the *data* of the problem).
2. Magnitudes that are not given, some, or all, of which have to be found.

Dynamical problems involve, generally, forces, lengths, and angles, some of which are given, and some of which have to be found from the given ones.

For example, consider the problem in Art. 28, fig. 39. Here there are three forces involved—*viz.* W, P, and N; and there are two angles, i and θ; and of these we have—

given things . .	W, i, θ;
not-given things . .	P, N.

Now, when we are asked to calculate P, we must observe that it would be ridiculous to present its value in terms of N. Thus, it is quite true that

$$P = N \frac{\sin i}{\cos (i + \theta)},$$

but as N is not given, and must itself be expressed in terms of the only force whose magnitude *is* given—*viz.* W,—this value of P gives us no useful information. If a force has to be calculated in any question, its magnitude must be expressed in terms of a *known* or *given* force—not in terms of a force whose magnitude is not given.

Consider, now, the question of Exercise 1, p. 54. Here we have—

given things	. .	P, Q, W, length AB;
not-given things	. .	pressures on pegs, position of equilibrium.

Now, in an actual example of such an arrangement there would be *two* other things given—*viz.* the length of the cord PAC between the mass P and the knot C, and also the length of the cord QBC. Why are these not mentioned in the data? Because they are useless. Since the mass of the cord is negligible, it is clear that when the system has settled into its position of equilibrium, the masses P and Q could each be attached to its cord higher up or lower down (by adding a new piece of cord), and this would not alter the equilibrium or the pressure on either peg. Hence it is evident that no use can be made, in the calculation, of the lengths PAC and QBC.

As regards the required position of equilibrium, the student should ask himself, at the outset, the question: "What unknown thing or things shall I take to define the position of equilibrium?"

Now we may seek either for the lengths AC and BC, or for the angles BAC and ABC; for, if we knew either these lengths or these angles, we should know the position into which the system settles.

31. **Funicular Polygons.—Simplest Cases.**—If we take a light, flexible cord, attach to it at various points a number of heavy bodies, and then hang up the system by fixing the ends of the cord, the figure formed by the cord is called a *Funicular Polygon.* A familiar instance occurs when a person has a number of apples or pears, partially ripe, which he wishes to ripen in the house. If they are tied by their stalks to a cord at points A_1, A_2, A_3, A_4, A_5 (fig. 42), and the ends, A_0, A_6, of the cord are then fastened to two fixed points, so that the whole

system hangs freely in a vertical plane, the portions A_0A_1, A_1A_2, A_2A_3, . . . between stalk and stalk may be considered as straight lines (since the weight of the cord is negligible), and they form a *funicular* (*i.e.* rope) *polygon*. By altering either the place of attachment or the weight of any one or more of the suspended bodies, the figure of the polygon is changed.

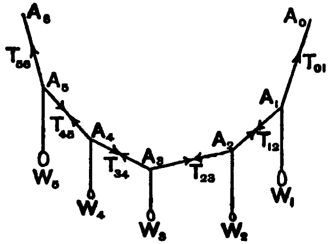


Fig. 42.

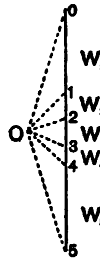


Fig. 43.

Let us now consider the equilibrium of the vertex A_1. It is kept at rest by three forces — *viz.* the weight, W_1, of the body suspended from it, the tension, T_{01}, of the cord, A_1A_0, and the tension, T_{12}, of the cord, A_1A_2. Consider the equilibrium of the vertex A_2. The forces keeping it at rest are W_2, the tension T_{12}, now acting in the sense A_2A_1, and the tension T_{23} of the cord A_2A_3.

Similarly for the equilibrium of the other vertices.

We can represent the relative magnitudes of these tensions and weights very easily thus: Along a vertical line, 05, fig. 43, take portions 01, 12, 23, 34, 45 proportional to the weights W_1, W_2, W_3, W_4, W_5; from 0 draw $0O$ parallel to A_0A_1, and from 1 draw $1O$ parallel to A_1A_2. Then, by the principle of the triangle of forces (Art. 20), the lengths

$$O0,\ O1,\ 01$$

are proportional, respectively, to the forces

$$T_{01},\ T_{12},\ W_1.$$

Now join O to 2, and we have a triangle O_{12} whose sides must represent the three forces T_{12}, W_2, and T_{23}, which keep the vertex A_2 at rest; for 12 represents W_1, and by the previous reasoning $O1$ represents T_{12}, therefore the third side, $O2$, must be parallel and proportional on the same scale to the third force T_{23}.

Similarly, O3 must be parallel to A_3A_4, and its length represents the tension, T_{34}, in A_3A_4; $O4$ must be parallel to A_4A_5,

and represent the tension in A_4A_5; and O5 must be parallel to A_5A_6, and represent its tension.

The figure (fig. 43) is called a *force diagram* of fig. 42, since the various lines in fig. 43 represent in magnitudes and directions the various forces in fig. 42.

It is very clear from fig. 43 that if the weights W_1, W_2, . . . of the suspended bodies are all given, and the directions of *any two* of the sides of the funicular in fig. 42 are known, the directions of all the other sides can be at once found. For, we have simply to take any vertical line, such as 05, and lay off successively on it lengths, 01, 12, . . . , representing W_1, W_2, . . . ; then, if the point O can be found, the directions of all the sides of the funicular are known. Suppose, for example, that the directions of the sides A_0A_1 and A_5A_6 are known; then from 0 and 5 in fig. 43 we draw 0O and 5O parallel to A_0A_1 and A_5A_6, and they will intersect in O; the lines joining O to the various points of division, 1, 2, 3, . . . , of the vertical line 05 will then be parallel to the various sides of the funicular in fig. 42.

There is one very important thing with regard to the various tensions, T_{01}, T_{12}, . . . that must be pointed out—viz. *though the tensions in the various parts of the cord may be all of different magnitudes, they have all the same horizontal component.*

This is obvious; because, if we consider the equilibrium of A_1, and resolve horizontally, we have simply the result—

horizontal component of T_{01} = horizontal component of T_{12}.

Similarly, if we consider the equilibrium of A_2, and resolve horizontally, we have—

horizontal component of T_{12} = horizontal component of T_{23};

and so on.

In fig. 43 the perpendicular from O on the line 05 represents the common horizontal component of the tensions (on the scale on which the W's are represented).

The student will do well to remember this *constancy of the horizontal component of tension* in any flexible cord or chain from any points of which heavy bodies are suspended, whatever the weights and positions of the bodies; it is also true for any heavy chain which hangs from two fixed points—the tension at every point has a constant horizontal component.

Take the simple case in which there are *two* suspended bodies.

Let A and B, fig. 44, be two points 20 inches apart in a horizontal line; let a light cord, $ACDB$, 34 inches long, have its ends attached to A and B; at C, 13 inches from A, let a mass of weight P be suspended; at D, 6 inches from C, let a mass of weight Q be suspended. What must be the ratio of P to Q so that the portion CD of the cord may be horizontal?

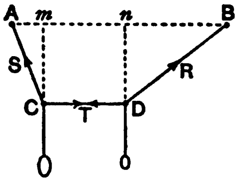


Fig. 44.

From C and D draw perpendiculars, Cm and Dn, to AB; then the lengths Am and Bn can be easily found; for, let $Cm = Dn = h$, let $Am = x$, $Bn = y$. Then—

$$13^2 = h^2 + x^2,$$
$$15^2 = h^2 + y^2,$$
$$\therefore\ y^2 - x^2 = 15^2 - 13^2 = 28 \times 2,$$
$$\text{and } y + x = AB - CD = 14,$$
$$\therefore\ y - x = 4,$$
$$\therefore\ y = 9,\ x = 5, \text{ and } h = 12.$$

Now, the *plan* on which we must proceed to obtain the result is simply this: Consider the equilibrium of the knot C; if T and S are the tensions in CD and CA, the forces keeping C in equilibrium are T, S, P. Similarly, if R is the tension in DB, the forces keeping D in equilibrium are T, R, Q; then for the equilibrium of C express T in terms of P, and for that of D express T in terms of Q; equating these two values of T, we shall have the relation between P and Q.

For the equilibrium of C, the triangle ACm is a triangle of forces—

$$\therefore\ \frac{T}{P} = \frac{Am}{mC} = \frac{x}{h} = \tfrac{5}{12};$$

for that of B, the triangle BDn is a triangle of forces;

$$\therefore\ \frac{T}{Q} = \frac{Bn}{nD} = \frac{y}{h} = \tfrac{9}{12};$$

hence we have $\qquad T = \tfrac{5}{12}P$, and $T = \tfrac{9}{12}Q$,

$$\therefore\ \frac{P}{Q} = \tfrac{9}{5}.$$

EXERCISES

1. In fig. 44, if $AB=5$ feet, $AC=17$ inches, $CD=16$ inches, $DB=39$ inches, find the ratio of P to Q so that CD may be horizontal.

Result. $$\frac{P}{Q}=\frac{9}{2}.$$

2. If $AB=40$ inches, $AC=17$, $CD=12$, $DB=25$, and $P=30$ pounds, find Q so that CD may be horizontal; find also the tensions in AC, CD, and DB.

Result. $Q=12$ pounds, $S=34$ pounds' weight, $T=16$, $R=20$.

If, as in fig. 44, a flexible cord, $ACDB$, has its ends, A and B, fixed at given points, while from two given points, C and D, in it are suspended two heavy bodies, and it is desired to have the direction of any one side of the figure $ACDB$ given, the requisite ratio of P to Q can be very easily found.

For example, in fig. 45 let the fixed ends A and B occupy any positions, let it be desired to have the side CD given in direction; then the figure $ACDB$ is given; for, from B draw Br equal to DC and in the given direction; then the point r is given; and since Cr is equal to DB (which is given in length), the three sides of the triangle ACr are given in magnitudes, therefore the point C is known, and thence the figure $ACDB$. Now, draw any vertical line, mp (fig. 46), take any point, O, and from O draw Om, On, Op parallel to DB, DC, CA; then

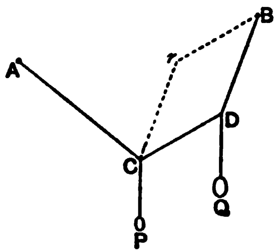


Fig. 45.

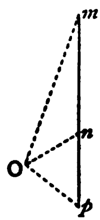


Fig. 46.

$$\frac{Q}{P}=\frac{mn}{np}.$$

This is obvious, because the vertex D is kept in equilibrium by Q and the tensions in DB and DC; and the sides of the triangle nmO are parallel to these; therefore, on the scale on which On represents the tension in CD, the weight

Q is represented by mn. Similarly, from the equilibrium of C, we find that P is represented on the same scale by np. Therefore, etc.

EXAMINATION ON CHAPTER IV

1. What is the definition of smooth surfaces?

2. State Newton's Third Axiom, or Law, of Motion.

3. If for every force in the Universe there exists an equal and opposite force, why can we not cancel every force by considering its opposite as well? (These opposites do not act on the *same body*.)

4. What is meant by the tension of a cord at any point in the cord?

5. What is the direction of the resultant pressure exerted on a smooth peg by a cord passing round it?

6. Define the principle of separate equilibrium or motion of any part of a given body or system of bodies.

7. What is the acceleration of a particle which is placed on a smooth inclined plane of 30° inclination? What is it for a smooth plane of any inclination?

8. If P and Q are given forces, a and b given lengths, α a given angle, and x a length to be found, criticise the result—

$$x = PQ\frac{a+b}{a^2}\tan\alpha.$$

State what is meant by the *homogeneity* of expressions.

9. What is meant by a funicular polygon?

10. Whatever the shape of the polygon formed by a cord from various points of which masses of any magnitudes are suspended vertically, what simple relation holds between the tensions in the various parts?

CHAPTER V

PARTICLES ON ROUGH PLANES

32. **Definition of Rough Surfaces.**—Two rough surfaces are such that when they are in contact at a point the resultant force exerted by one on the other need not act along the normal to their surface of contact.

Consider the two surfaces in contact at the point P in fig. 32, p. 45. Suppose AB to be a table, and GP to be a pencil of wood, so light that its weight can be neglected. We know by experience that it is possible to press the pencil in the direction, GP, of its length against the table, and to incline it at a considerable angle to the normal before it slips along the table at P. Let us suppose that the pencil is pressed with any force in the line GP, and account for its equilibrium in this inclined position. It is obviously kept at rest by two forces only—viz.:

(*a*) the applied pressure in GP,
(*b*) the reaction of the table at P.

Now, for equilibrium these two forces must be equal and opposite in the same right line (Art. 21); hence the table must be reacting in the line PG—*i.e.* in a direction inclined to the normal. We can, perhaps, incline the pencil more and more to the normal, until we reach such a deviation from the normal that equilibrium can no longer continue, and the pencil slips. Hence there is a limit to the angle which the reaction of the table can make with the normal. The greatest angle that this reaction can make with the normal is called

the angle of friction

between the two bodies.

This angle of friction depends on the natures of the two surfaces in contact. Thus, if the pencil is made of cedar wood and the table of oak, the angle of friction may be (say) 25°.

If the same pencil is used, and the oak table is replaced by a plate of glass, it may be found that the reaction cannot be inclined at an angle of more than 15° to the normal; that is, the angle of friction between the cedar wood and the glass is 15°. (These figures are not to be supposed to be *accurate*; they are probably somewhere in the neighbourhood of the truth; but they would require to be found by actual experiment.) Experiments have been made with many pairs of bodies for the purpose of measuring the angles of friction between them, and it is found that two bodies have to be very rough indeed to give an angle of friction as great as 45°. Practically we may consider that no angle of friction is as great as 45°—although a very few such have been observed.

There is a slightly different way of presenting this result.

Let fig. 47 represent the pencil at any inclination to the normal, the pressure, R, applied by the finger at G, acting in GP, so that the table reacts along PG with a force equal to R. We can resolve the reaction, R, of the table into a normal component, N, and a component, F, along the table. This component, F, along the surface of contact could not exist, of course, if the bodies were smooth. It is called the *force of friction* between the bodies.

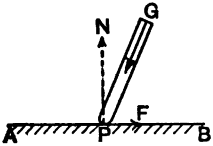


Fig. 47.

Another simple experiment illustrates the same state of things.

Suppose that AB (fig. 48) is a rough plane, having a body of weight W with a flat base resting on it, the plane AB being horizontal. The body is in equilibrium under the action of only two forces—viz. W and the reaction, R, of the plane. Hence R must be, in fig. 48, a vertical upward force equal to W, and it is normal to the plane. In fig. 48 what is the force of friction between the body and the plane? There is no friction here called into play.

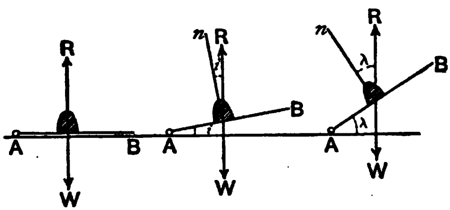


Fig. 48. Fig. 49. Fig. 50.

Suppose that the plane is slightly tilted up, as in fig. 49, and that the body still rests. Then, as before, since the body is kept at rest by only two forces—viz. W and the reaction, R, of the plane, R must still be vertical and equal to W. Is there friction called into play in fig. 49? Yes; because R is now inclined to the normal n, and if i is the inclination of the plane to the horizon, the normal component of R is $R \cos i$—i.e. $W \cos i$, and the tangential component, or component along the plane, is $W \sin i$. This latter is the force of friction, and observe that it is acting in a direction opposite to that in which the body is trying to slip. The body is trying to slip *down* the plane, because if we imagine the plane in fig. 49 to become gradually smoother and smoother, the body would eventually slip down.

Suppose that we still further increase the inclination of the plane, until we can increase it no more (fig. 50) without causing the body to slip down; then the inclination of the plane is the angle of friction between the body and the plane, because R is still equal and opposite to W, and its inclination to the normal n is equal to the inclination of the plane, so that R is now making the greatest possible angle with the normal. This angle must, therefore, be the angle of friction, and we have denoted it by the Greek letter λ in the figure.

Hence we get an apparently new definition of the angle of friction between two bodies—viz. *the angle of friction between any two bodies,* A *and* B, *is the greatest inclination of a plane made of one of the bodies* (A) *on which a slab of the other body* (B) *would rest, when acted on by no force but its own weight and the reaction of the plane.*

(For this experiment the body B must have a flat base cut on it, and not a round one, so that it cannot roll; and it must not be able to topple over, so that the form of a slab is best.)

It was thus—by measuring the greatest angle of tilt of an inclined plane—that Coulomb measured the angles of friction for various bodies.

In fig. 49, if the force of friction is F, and the normal pressure N, we have—

$$F = W \sin i,$$
$$N = W \cos i;$$
$$\therefore \frac{F}{N} = \tan i.$$

Now the greatest value of i is λ, so that the greatest value of $\frac{F}{N}$ is $\tan\lambda$.

DEF.—The greatest ratio that the force of friction can bear to the normal pressure between two bodies is called *the coefficient of friction* between them.

We thus see that the coefficient of friction is the tangent of the angle of friction.

This coefficient of friction we shall denote by the Greek letter μ.

Suppose, for example, that the coefficient of friction between two bodies is $\frac{1}{2}$. Does this imply that in *every* case in which these bodies rest against each other at a point $F=\frac{1}{2}N$, where F and N are the force of friction and the normal pressure between them? No; not by any means. Thus, the coefficient of friction between the table on which we write and a certain piece of stone may be $\frac{1}{2}$; but if we place the stone on the table, the force of friction is actually nothing at all, while the normal pressure is equal to the weight of the stone. What is meant is this: if the stone is, owing to any cause, *just about to slip* on the table, F will be equal to $\frac{1}{2}N$.

In general, if μ is the coefficient of friction between two bodies, the *greatest* value of $\frac{F}{N}$ is μ; and $F=\mu N$ *only when slipping is about to take place.* If slipping is not on the point of taking place, F is always less than μN.

Observe, then, that if a body which is acted upon by no force but its weight is placed on a rough inclined plane, the body will rest if the inclination of the plane is less than the angle of friction, and it will slide down if the inclination is greater than the angle of friction.

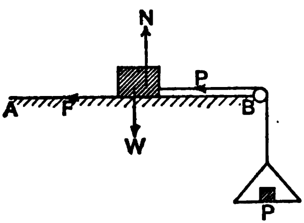


Fig. 51.

There is another, and more simple, method of measuring the coefficient of friction between two bodies.

Let AB (fig. 51) be a horizontal plane made of one of the bodies, and let the other body, whose weight is W, be placed

on this plane; let this body be attached to a light flexible cord which, passing over a fixed smooth pulley at B, sustains a scale-pan which can be loaded to any amount. Suppose that the scale-pan is loaded until the body is just about to slip on AB, and let P be the total load; then we may consider the body as kept in equilibrium by four forces — viz. its weight, W; the tension, P, of the cord; the normal pressure, N, of the plane; and the force of friction, F, which acts in a sense to oppose the motion.

Hence, resolving vertically and horizontally—

$$N = W,$$
$$F = P;$$
$$\therefore \frac{F}{N} = \frac{P}{W} = \mu,$$

remembering that we have assumed P to be such that slipping is on the point of taking place. Observe that for all values of P from zero up to the greatest value, we have $F = P$, while, of course, $N = W$; but $\frac{F}{N}$ is not equal to μ until slipping is about to take place. This method was that used by General Morin.

A few examples will help to a clearer understanding.

EXAMPLES

1. A mass of 390 pounds is placed on a rough inclined plane, AB (fig. 52), inclined at an angle, i, whose tangent is $\frac{5}{12}$, to the horizon, the coefficient of friction between the body and the plane being $\frac{1}{3}$; a cord is attached to the body, and pulled in a direction making an angle θ with the upper side of the plane, such that $\tan\theta = \frac{3}{4}$; find the tension of this cord, which—

(*a*) will just drag the body up,
(*b*) will just prevent the body from sliding down.

Find also the value of the normal pressure on the plane in each case.

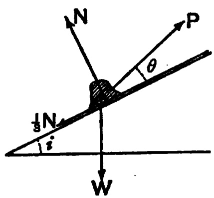


Fig. 52.

Taking (*a*) first, the body is just about to slide *up*, therefore the force of friction acts *down* the plane. Also, since slipping is about to take place, the force of friction $= \frac{1}{3}N$, where N is the the normal pressure of the plane against the body.

Now, resolving forces along the plane for the equilibrium of the body, we have—

$$P\cos\theta - \tfrac{1}{3}N - 390\sin i = 0,$$
$$\text{or}\quad \tfrac{4}{5}P - \tfrac{1}{3}N - 150 = 0 \qquad . \quad . \quad . \quad . \quad . \qquad (1)$$

Resolving perpendicularly to the plane,

$$P\sin\theta + N - 390\cos i = 0,$$
$$\text{or}\quad \tfrac{3}{5}P + N - 360 = 0 \qquad . \quad . \quad . \quad . \quad . \qquad (2)$$

Eliminating N from (1) and (2), we have—

$$P = 270 \text{ pounds' weight};$$

and we have also—

$$N = 198 \quad ,, \quad ,,$$

Take now case (*b*). Since we now suppose that slipping is just about to take place *downwards*, the force of friction acts *upwards*, and if the normal pressure is now N', the force of friction is $\tfrac{1}{3}N'$; so that resolving along and perpendicular to the plane, we have now, if P' is the tension—

$$\tfrac{4}{5}P' + \tfrac{1}{3}N' - 150 = 0 \qquad . \quad . \quad . \quad . \quad . \qquad (3)$$
$$\tfrac{3}{5}P' + N' - 360 = 0 \qquad . \quad . \quad . \quad . \quad . \qquad (4)$$

which give—

$$P' = 50 \text{ pounds' weight},$$
$$N' = 330 \quad ,, \quad ,,$$

Hence, when the tension of the cord is 270 pounds' weight, the body is on the point of being dragged up, and the friction acts downwards; when the tension is 50, the body is on the point of sliding down, and the friction acts upwards. It is evident, therefore, to common-sense, that there must be some value of the tension for which there will be *no friction* whatever called into play, and the plane is behaving as if it were smooth.

2. In the last, find the tension of the cord when there is no friction called into play, and find the corresponding normal pressure.

If P is the tension and S the normal pressure, the force of friction being zero, we have, by resolving along and perpendicularly to the plane,—

$$\tfrac{4}{5}P = 150, \quad \therefore\ P = 187\tfrac{1}{2} \text{ pounds' weight},$$
$$\tfrac{3}{5}P + S - 360 = 0, \quad \therefore\ S = 247\tfrac{1}{2} \quad ,, \quad ,,$$

Thus the values of the tension and the normal pressure in this case are intermediate to the values in the extreme cases of the last example.

3. Two masses, 6 pounds and 8 pounds, are connected together by a tight cord and placed on a rough inclined plane whose inclination is gradu-

ally increased; the coefficients of friction for these bodies and the plane are $\frac{1}{3}$ and $\frac{1}{2}$ respectively; find the greatest inclination of the plane—

(*a*) supposing that the 6 lb. mass is the lower,
(*b*) „ „ 8 „ „

Suppose fig. 53 to represent case (*a*). Now, when the inclination reaches the value $\tan^{-1}\frac{1}{3}$, the lower mass tries to slip, and it would slip if it were not hampered by its connection with the other mass. The upper mass if entirely alone would not slip until the inclination became $\tan^{-1}\frac{1}{2}$. The result is that the inclination at which the two slip together is neither of these, but

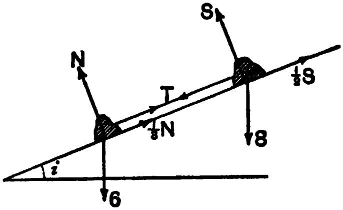


Fig. 53.

some intermediate value—suppose i. Let T be the tension of the cord; then the forces acting on the lower are the normal pressure N, the force of friction, $\frac{1}{3}N$, acting upwards, T, and its weight; so that for its equilibrium we have—

$$N=6\cos i;\quad T+\tfrac{1}{3}N=6\sin i\quad \therefore\ T=6\sin i-2\cos i.$$

Similarly, if S is the normal pressure on the upper mass, we have for it—

$$S=8\cos i;\quad \tfrac{1}{2}S=T+8\sin i\quad \therefore\ T=4\cos i-8\sin i;$$

and, equating the values of T, we have—

$$\tan i=\tfrac{3}{7},$$

for the slipping inclination.

4. If in the last question the positions of the masses are interchanged, so that the 8 lb. is the lower, while the two are still connected by a cord, find the inclination at which slipping takes place.

Here, as before, the 6 lb. mass tries to slip when $\tan i=\frac{1}{3}$; and now there is nothing to prevent this, because the cord cannot *push* against the body; hence, when $\tan i=\frac{1}{3}$ the upper mass will slip, the lower one not moving.

5. If the *cord* connecting the two masses in the previous questions is replaced by a wire or thin bar, what will the limiting inclination be?

In this case it makes no difference which body is the lower: neither can slip without causing the other to move, and the slipping inclination is that given in Example 3.

6. If in Example 1 the cord is pulled with a tension of 335 pounds' weight, all the other data being the same as before, find the acceleration produced in the body.

In this case let X be the total component force acting on the body in the up-plane direction, where N is the normal pressure; then—

$$\begin{aligned} X &= 335 \times \tfrac{4}{5} - \tfrac{1}{3}N - 390 \times \tfrac{5}{13} \\ &= 268 - \tfrac{1}{3}N - 150 \\ &= 118 - \tfrac{1}{3}N. \qquad (1) \end{aligned}$$

But since there is no motion of the body in the direction of the normal, the total normal component is zero. Hence, resolving along the normal,

$$335 \times \tfrac{3}{5} + N - 390 \times \tfrac{12}{13} = 0$$

$$\therefore N = 159 \qquad (2)$$

Substituting this value of N in (1), we have $X = 65$ pounds' weight; so that by (4), p. 70,

$$\frac{a}{g} = \frac{65}{390}$$

$$\therefore a = 5\tfrac{1}{3} \ {}^{f}/_{ss}.$$

7. If in last example the cord is pulled with a force of 400 pounds' weight, what is the acceleration of the body? What is the normal pressure on the plane?

Ans. $a = g = 32 \ {}^{f}/_{ss}$; and $N = 120$ pounds' weight.

8. In the same example find the greatest value of the tension that will allow the body to stay on the plane.

Result. 600 pounds' weight.

9. What is the acceleration of a particle which is sliding down a rough plane whose inclination is i, the coefficient of friction being μ?

Ans. $a = g(\sin i - \mu \cos i)$.

10. Explain why this is negative if $\tan i < \mu$.

EXAMINATION ON CHAPTER V

1. Define rough surfaces.
2. What is meant by the *angle of friction* between two rough surfaces?
3. How can this angle be measured by means of an inclined plane?
4. Define the *coefficient* for two rough surfaces. Has any *one* body a coefficient of friction?
5. When can the force of friction between two rough bodies be put equal to the normal pressure multiplied by the coefficient of friction?
6. What direction in the tangent plane is assumed by the force of friction?
7. Describe the methods of Coulomb and Morin for finding coefficients of friction.

CHAPTER VI

UNIFORMLY ACCELERATED RECTILINEAR MOTION

33. **Relations between Time, Velocity, and Distance travelled.**—If a point moves in such a way that a feet per second are added to its velocity at the end of every second, in t seconds the increase of its velocity will be at feet per second; so that if u feet per second is the velocity at the beginning of the time t, and v is the velocity at the end of this time,

$$v = u + at. \quad . \quad . \quad . \quad . \qquad (1)$$

As a numerical example, suppose that the velocity of a point is at the present moment $60\,{}^f/_s$, and that a uniform increase of $10\,{}^f/_s$ is made every second; then in 6 seconds the total addition to the original velocity is 10×6, or 60, feet per second, so that the velocity is $120\,{}^f/_s$ at the end of 6 seconds.

If, on the other hand, there is a constant *decrease* of $10\,{}^f/_s$ every second, the velocity becomes 50, 40, 30, . . . feet per second at the end of 1, 2, 3, . . . seconds. At the end of 6 seconds the velocity $= 60 - 10 \times 6 = 0$; at the end of 7 seconds the velocity is $-10\,{}^f/_s$—*i.e.* $10\,{}^f/_s$ in a sense opposed to that of the original velocity.

When velocity is constantly diminished, the motion is said to be *retarded*. In the second case above there is said to be a uniform retardation of $10\,{}^f/_{ss}$. Of course, there is no essential difference between *acceleration* and *retardation*; retardation of a velocity means simply acceleration in the sense opposite to that of the velocity.

It is obvious, then, that according as a is in the same sense as v or in the opposite, we have—

$$v = u \pm at, \quad . \quad . \quad . \quad . \qquad (2)$$

where u is the velocity of the moving point at *any* instant, and

v the velocity at the end of t units of time after that instant, and in the same sense as that of u.

There is, however, no necessity for departing from equation (1), if we understand that in it v, u, and a are *all measured in the same sense*; on this understanding (2) is included in (1).

Velocity Diagram.—A diagram which represents the magnitudes of the velocities of a moving point at various times is sometimes called a *velocity-time curve.* To understand its nature, let us take the case in which a point is moving with a constant acceleration of $4\,{}^{f}/_{ss}$, and let its velocity at a given instant be $20\,{}^{f}/_{s}$.

Let OV and OT, fig. 54, be two perpendicular lines, each divided into a large number of equal segments. (The segments of the one need not be equal to those of the other, although they are so in the ordinary section-paper on which diagrams are drawn.) Now, let each division along OT represent a time of 1 second, and let each division along OV represent a velocity of $10\,{}^{f}/_{s}$. We may, if we please, take a division along OT to represent 2 or 5 or any number of seconds, and similarly a division along OV to represent a velocity of any number of feet per second; but we shall take the above numbers, as they are convenient enough for our numerical case.

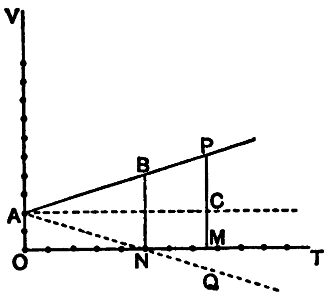


Fig. 54.

What is the velocity at the end of 5 seconds? By (1) above it is $20+4\times5$, or $40\,{}^{f}/_{s}$. Now, the time $t=0$ is represented by the point O, and at this time $v=20$, so that if we take $OA=$ two divisions along OV, OA represents the velocity at the beginning. When $t=5$, $v=40$; take, then, $ON=5$ divisions along OT, and draw NB perpendicular to ON, and equal to 4 divisions of OV, and we have the velocity at the end of 5 secs. represented by the ordinate BN. Similarly, if OM represents any number, t, of time units, and we erect MP perpendicular to OM and representing $20+4t$, we get the point P, whose ordinate, PM, represents v.

The curve $ABP\ldots$, formed by the points whose ordinates represent the velocities of the moving point at various times, is called the *velocity-time curve*, or sometimes the *velocity diagram*, the first being the more accurate term.

In the case of constant acceleration this curve is a right line; because, if we draw AC parallel to OT to meet MP in C, we have $MP = MC + CP = 20 + 4t$, $\therefore$ $CP = 4t$—*i.e.* PC is proportional to CA, $\therefore$ etc.

If the original velocity, instead of being *accelerated*, is *retarded* at the rate of $4\ {}^{f}/_{ss}$, the velocity at the end of 5 secs. is $20 - 4 \times 5$ —*i.e.* zero; hence, taking $ON = 5$ units, the velocity-time curve is the right line, ANQ, and the velocity becomes negative—*i.e.* reversed in direction—after $t = ON$.

If the acceleration is not constant, it is obvious that the velocity diagram is no longer a right line—it may be any curve whatever.

The line OT is called the *time-axis*, and OV is called the *velocity-axis*.

In the case of a point which moves with *constant* velocity (*i.e.* no acceleration), the distance gone over in any time is obtained by multiplying the velocity by the time. For instance, if a train is moving with a constant velocity of 25 miles per hour, the distance travelled in 3 hours is 25×3 miles. Generally if v is a *constant* velocity, the distance, s, travelled in t units of time is given by the equation—

$$s = v.t. \qquad . \quad . \quad . \quad . \quad (3)$$

When a point moves with a velocity which is perpetually changing, we cannot, of course, connect the distance travelled and the time by such an operation as (3).

How, then, can we find the distance travelled in any given time in such a case? By means of the following principle: *However variable the velocity of a moving point may be, it can be regarded as constant during an infinitesimally small time.*

This very simple principle is the very essence and foundation of the Integral Calculus.

Suppose that the curve whose ordinates represent the various velocities of a moving point is $APQC$ (fig. 55); let the time-axis, OT, be divided into an indefinitely-great number of small elements, MR, RS, etc. At the time represented by OM, the velocity of the moving point (which point, be it

observed, is not at all represented in the figure) is represented by MP, and in the interval of time MR, the distance travelled by the moving point, is represented by $MP \times MR$ (since the equation $s = v.t$ holds for the time MR)—*i.e.* by the area of the little rectangle $PMRQ$. Similarly, the velocity throughout the time RS may be taken as constant, and represented by the ordinate RQ, so that the distance travelled in the time RS is represented by the area of the little rectangle QS; and so on. Hence it is obvious that the distance gone over by the moving point from the time represented by OM to the time represented by OD is represented by—

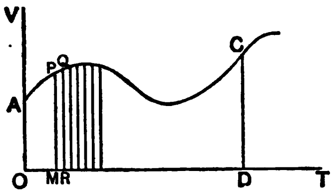


Fig. 55.

the area $PMDC$ of the velocity diagram.

This area is, of course, a number of square inches, or square feet, and it must be translated into distance (linear feet or inches) when we know the scales on which time and velocity are represented along the time-axis OT and the velocity-axis OV.

Take, now, the case of *constant acceleration*. The velocity diagram is a right line, AP (fig. 54); and if OM represents any time, t, and u (represented by OA) is the velocity of the moving point when $t = 0$, we have—

$$MP = u + at;$$

and the distance gone over in time t is represented by the area of the trapezium $OMPA$, which = area of rectangle $OMCA$ + area of triangle $ACP = OA \times OM + \frac{1}{2} CP \times AC$. Hence, if s is the distance moved over,

$$s = ut + \tfrac{1}{2} at^2, \quad . \quad . \quad . \quad . \quad (4)$$

in which s, u and a, are all supposed to be measured in the same sense.

To find the relation between the velocity, v, of the moving point when it has gone over the distance, s, and the velocity, u, at the beginning of s, eliminate t from (4) and (1), and we have—

$$v^2 = u^2 + 2as, \quad . \quad . \quad . \quad . \quad (5)$$

in which s, v, u, a are all measured in the same sense: in (5), however, the senses of v and u are of no consequence.

Collecting our results for uniformly-accelerated motion, we have—

$$v = u + at, \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$
$$s = ut + \tfrac{1}{2}at^2, \quad . \quad . \quad . \quad . \quad (\beta)$$
$$v^2 = u^2 + 2as. \quad . \quad . \quad . \quad . \quad (\gamma)$$

A numerical example will be advisable. Let the velocity be $20\,^{f}/_{s}$ at a given instant, and the acceleration $4\,^{f}/_{ss}$ in the opposite sense; let $A'A$ be the line of motion of the moving point (fig. 56), and O the initial position of this point, its velocity being here in the sense OA. What is the velocity, and what is the distance from O at the end of 3 seconds?

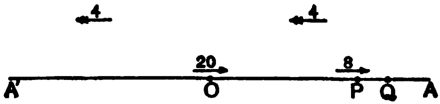


Fig. 56.

The velocity is to be got from (α), and it $= 20 - 4 \times 3 = 8\,^{f}/_{s}$, so that the point is still moving in the sense OA. The distance from O is got from (β), and it $= 20 \times 3 - \tfrac{1}{2}.4.9 = 42$ feet. If, then, P is the position of the moving point at the time $t = 3$, we have $OP =$ 42 feet.

What are the velocity and distance at the end of 5 seconds? From (α), $v = 20 - 4 \times 5 = 0$, and $s = 20 \times 5 - \tfrac{1}{2}.4.25 = 50$ feet; so that if we take $OQ = 50$, Q is the position of the moving point, and its velocity at Q is zero: the point is about to move back to O.

When will the point be back at O, and what velocity will it have at O?

When it is back at O, s is zero; put $s = 20t - \tfrac{1}{2}.4.t^2 = 0$, and we get either $t = 0$ or $t = 10$; the first value of t is, of course, obvious; the second shows that after 10 seconds the moving point has returned to O. The velocity is obtained from (α), and we have—

$$v = 20 - 4 \times 10 = -20\,^{f}/_{s},$$

i.e. the velocity at O is the same as it was originally, but in the opposite direction.

What happens after $t = 10$ seconds? The moving point continues to move in the direction OA' with constantly-in-

creasing velocity. Thus, when $t=12$ secs. (γ) gives $s=-48$ feet—*i.e.* the moving point is 48 feet from O towards the left of the figure.

Observe that in equations (α), (β), (γ) the velocities u and v are *any two velocities whatever* of the moving point, the time interval between them being t, and the space interval s.

For example, let us take the two velocities belonging to the points Q and P in fig. 56, and apply (γ); then we have—

$$0^2=8^2-2\times4\times PQ;\ \therefore\ PQ=8,$$

which we know to be true.

34. **Falling Particle.**—If a particle is projected vertically upwards from the Earth's surface with a velocity V, what height will it attain?

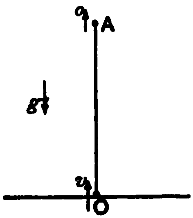


Fig. 57.

The diagram representing this case is fig. 57; O is the point of projection, the velocity at which is V; A is the highest point attained, the velocity at which is 0; hence, using (γ) of last article,

$$0^2=V^2-2g.OA$$

$$\therefore\ h=\frac{V^2}{2g}, \quad . \quad . \quad . \quad (1)$$

where $h=OA$. The "height due to the velocity V" is an expression employed to denote the greatest height to which a particle will attain if it is projected vertically upwards with the velocity V; we see that it is $\frac{V^2}{2g}$.

If a particle falls from rest through a height, h, what velocity will it acquire?

The same diagram will do, reversing the sense of the arrow at A, and we have from (γ)—

$$V^2=0^2+2g.OA$$

$$\therefore\ V=\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad (2)$$

where $h=OA$. This velocity is often spoken of as the "velocity due to a height h."

If t is the number of seconds taken by a particle to fall from rest through a height h feet we have—

$$h = \tfrac{1}{2}gt^2 = 16t^2 = (4t)^2, \qquad . \qquad . \qquad (3)$$

taking $g = 32\ ^f/_{s^2}$. Now, $4t$ is the number of quarter-seconds taken to fall, so that (3) gives the rule for calculating the height of a tower (or the depth of the surface of water in a well),

the square of the number of quarter-seconds

taken by a stone to fall is the height in feet.

EXAMPLES

1. A and B (fig. 58) are two points in the same vertical line, B being 240 feet above A; from B a particle is projected upwards with a velocity of 120 $^f/_s$, and 1 second afterwards a particle is projected upwards from A with a velocity of 174 $^f/_s$; when and where will they meet?

Suppose that they meet at P, and that the first particle has been moving for t seconds; then by (β), p. 77,

$$PB = 120t - 16t^2. \qquad . \qquad . \qquad . \qquad (1)$$

The second particle has been moving for $t - 1$ seconds; therefore

$$PA = 174(t-1) - 16(t-1)^2. \qquad . \qquad . \qquad (2)$$

Now, $PA - PB = 240$, $\therefore$ from (1) and (2) we get—

$$t = 5 \text{ seconds.}$$

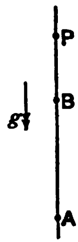


Fig. 58.

This gives $PB = 200$ feet.

2. If in the last question the velocities of the particles projected from B and A are 60 $^f/_s$ and 99 $^f/_s$ respectively, the other data remaining unaltered, when and where will they meet?

Supposing them to meet at P (fig. 58), we have—

$$PB = 60t - 16t^2, \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (3)$$
$$PA = 99(t-1) - 16(t-1)^2 \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (4)$$
$$\therefore \quad t = 5.$$

Equation (3) gives, then, $PB = -100$—i.e. P is *between* B and A, and 100 feet below B.

3. In the same question if the velocities of the particles projected upwards from B and A are 20 $^f/_s$ and 49 $^f/_s$, when and where will they meet?

Ans. 5 seconds after the projection of the particle from B, and at a point 60 feet *below* A.

4. A point, moving in a right line from a position, O, with constant acceleration, is observed to move over $16\frac{3}{4}$ feet in the 4th second from the instant of leaving O, and to move over $18\frac{3}{4}$ feet in the 8th second; find the acceleration and the velocity in the position O.

Let A be its position when $t=3$ seconds, and B when $t=4$: the interval between these times is the 4th second. Let u be the velocity at O, and α f/ss the acceleration in the sense OAB; then—

$$OA=3u+\tfrac{1}{2}\alpha.9,$$
$$OB=4u+\tfrac{1}{2}\alpha.16,$$
$$\therefore\ 16\tfrac{3}{4}=u+\tfrac{7}{2}\alpha. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, let P be its position when $t=7$, and Q when $t=8$;

$$\therefore\ OP=7u+\tfrac{1}{2}\alpha.49$$
$$OQ=8u+\tfrac{1}{2}\alpha.64$$
$$\therefore\ 18\tfrac{3}{4}=u+\tfrac{15}{2}\alpha. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we obtain—

$$u=15 \text{ f/s and } \alpha=\tfrac{1}{2} \text{ f/ss}.$$

5. If in the last the distances travelled in the 4th and 8th seconds are $29\frac{1}{2}$ feet and $17\frac{1}{2}$ feet respectively, find the acceleration and the velocity at O.

Result. The velocity at O is 40 f/s, and the acceleration is 3 f/ss, in the opposite sense.

6. If the distance gone over in the t^{th} second is x, and the distance gone over in the $(t+n)^{\text{th}}$ second is y, show that the acceleration is

$$\frac{y-x}{n}.$$

7. If the distances gone over by a point moving with uniformly-accelerated motion in the p^{th}, q^{th}, and r^{th} seconds are, respectively, x, y, and z, show that—

$$(q-r)x+(r-p)y+(p-q)z=0.$$

8. A particle is projected vertically upwards from the ground with a certain velocity, and, after reaching a height of 576 feet, it takes 5 seconds to return to this height; find the velocity of projection.

Let A be the point of projection and B the point 576 feet above it (fig. 58); then if u is the velocity at A, the velocity at B is $\sqrt{u^2-2g\times 576}$. Now, take B as a starting B, and if t is reckoned from B, the distance, s, of the moving point from B at the end of t seconds is given by the equation (see p. 78)

$$s=t\sqrt{u^2-2g\times 576}-\tfrac{1}{2}gt^2,$$

s being measured in the (upward) sense of the velocity at B. Take $g=32$; then we are given that $s=0$ when $t=5$; therefore—

$$0=\sqrt{u^2-64\times 576}-16\times 5,$$
$$\therefore\ u=208 \text{ f/s}.$$

9. A particle is projected upwards with a certain velocity, and after reaching a height of 225 feet it takes 10 seconds to return to this height; find the velocity of projection.

Result. 200 $^{f}/_{s}$.

10. If a particle is projected upwards with any velocity, and P is any point in its line of motion, show that the velocity with which the particle returns to P in the downward motion is equal to that which it had at P in the upward motion.

11. A particle starts from O along a right line with a velocity of 10 $^{f}/_{s}$ and an acceleration of 4 $^{f}/_{ss}$ in the same sense; after 5 seconds the acceleration becomes 3 $^{f}/_{ss}$ in the opposite sense; how long will its motion in the original sense continue, and how far from O will it be when its velocity in the original sense ceases?

Ans. It will be moving in the original sense for 15 seconds altogether; distance from $O = 250$ feet.

12. A particle is dropped from a height of 196 feet from the ground, and at the same instant another particle is projected vertically from a point 56 feet above the ground; with what velocity must the second particle be projected so as to reach the ground at the same instant as the first?

Ans. An *upward* velocity of 40 $^{f}/_{s}$.

13. A stone is dropped into a well, and in n seconds the sound of the splash is heard; assuming that the velocity of sound in air is 1092 $^{f}/_{s}$, find the depth of the well.

Let t be the number of seconds taken by the stone to fall, and x the depth of the surface of the water in feet; then $t = \sqrt{\frac{2x}{g}} = \frac{\sqrt{x}}{4}$. Also if t' is the time taken by the sound to reach the top, $t' = \frac{x}{1092}$. Now we are given that $t + t' = n$; therefore—

$$\frac{x}{1092} + \frac{\sqrt{x}}{4} = n,$$

a quadratic for $\sqrt{x}$.

14. In the last, if the sound of the splash is heard after 6 seconds, what is the depth?

Ans. 489 feet, very nearly.

15. O is a point on a smooth horizontal table 55 inches from the edge, along which there is fixed a vertical plane; at the same instant two particles are projected from O along the same right line, which is perpendicular to the edge, one particle with a velocity of 20 inches per second, and the other with a velocity of 12 inches per second; if, on reflection from the edge, a particle returns with half its original velocity, where will the two particles encounter each other?

Ans. 10 inches from the edge.

16. If in the last the particle whose velocity is 12 inches per second is projected $\frac{1}{4}$ of a second after that whose velocity is 20 inches per second, where will they meet?

Ans. 15 inches from the edge.

17. If the distance between O and the edge of the table is h feet, and a particle is projected from O with a velocity of u f/s, while it is accelerated towards the edge with α f/ss, and when it strikes the edge is reflected with a velocity which is e times the velocity of impact, find how far it will move from the edge before losing its velocity, the acceleration towards the edge always continuing.

Result. $$e^2\left(\frac{u^2}{2\alpha}+h\right).$$

18. If in the last the particle again strikes the edge, is again reflected, and so on, find the successive distances from the edge at which the motion stops for an instant.

Result. The distances are $\frac{u^2}{2\alpha}+h$ multiplied by e^2, e^4, e^6, etc.

35. **Motion of two connected Particles.**—Let P and Q (fig. 59) be two particles, whose weights we shall also designate by P and Q, connected by a fine, inextensible, and perfectly flexible cord, whose weight is negligible, which passes over a fixed pulley, C, supposed to be devoid of friction; and let it be required to find the motion of each particle and the tension of the cord.

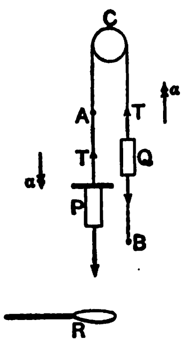


Fig. 59.

There being no friction or weight of cord, the tension at a given instant will be the same throughout the cord. Denote this by T. Also, since the cord is inextensible, by whatever amount the length of CP is increased the length of CQ is diminished; that is, the velocity of P is at each instant the same in magnitude as the velocity of Q; and, therefore, the acceleration, α, of P is also the same in magnitude as that of Q, but one will be downwards and the other upwards. (P is assumed to be greater than Q.) Now, the total downward force acting on P is $P-T$, and the total upward force on Q is $T-Q$; hence the equations of motion of P and Q are—

$$\frac{P-T}{P}=\frac{a}{g}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{T-Q}{Q}=\frac{a}{g}, \quad . \quad . \quad . \quad . \quad (2)$$

which give—

$$a=\frac{P-Q}{P+Q}.g, \quad . \quad . \quad . \quad (3)$$

$$T=\frac{2PQ}{P+Q}; \quad . \quad . \quad . \quad (4)$$

and these show that both a and T are constant, T being the *harmonic mean between the weights.* If P has moved from rest through a distance, AP, or x, we have for the time, t, of the motion $x=\frac{1}{2}at^2$ (Art. 33); hence—

$$x=\frac{1}{2}\frac{P-Q}{P+Q}gt^2. \quad . \quad . \quad . \quad (5)$$

Equation (5) shows that such an arrangement enables us to find the value of g, and for this purpose the arrangement was employed, in principle, in *Atwood's Machine.* Simpler and more accurate ways of measuring g than that furnished by this old machine have, however, been recently used, so that Atwood's machine may be regarded as a mere historical curiosity—interesting, however, for the ingenious notion which it embodies. The notion is this: We might calculate the value of g by measuring the time taken by a particle falling freely near the Earth's surface to move from rest through a measured height, h; for this purpose our equation would be $h=\frac{1}{2}gt^2, \therefore g=\frac{2h}{t^2}$. But a free particle falls too rapidly for accurate observation—in 1 second it falls through 16 feet—and, moreover, unless it is enclosed inside a glass tube deprived of air, the resistance of the air will vitiate the experiment. Hence, if we can obtain a case of motion in which the acceleration is only a *small* and *known* fraction of g, measurement becomes much more easy. Now, in the above arrangement, by selecting for P and Q two very nearly equal masses, the acceleration with which we have to deal—*viz.* $\frac{P-Q}{P+Q}g$, is a very small and known fraction of g, so that the time of falling is easily measured, and thence the value of g obtained from equation (5), which gives

$$g=\frac{P+Q}{P-Q}\cdot\frac{2x}{t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Atwood's machine is sometimes expressively described as "machine for diluting g." The value of g in this arrangement can be found by a different observation—thus: Let A be the initial position of P; at a measured

distance, h, below A let there be fixed a horizontal ring, R, through which P can pass freely; if on the top of P there is laid horizontally a small bar of metal longer than the diameter of the ring, this bar would be picked off by the ring when P is passing through. Now let P consist of a mass exactly equal to Q, together with this little bar whose weight is w; then $P-Q=w$, and $P+Q=2Q+w$;

$$\therefore\ \alpha=\frac{w}{2Q+w}.g, \qquad \text{(7)}$$

and if v is the velocity of P just as the bar is being lifted off by the ring, we have

$$v^2=\frac{2w}{2Q+w}.gh, \qquad \text{(8)}$$

in which equation we know w, Q, and h. Now, when the bar is picked off, the two masses moving become Q and Q, and in their motion the acceleration is zero, so that their velocities remain constant, each equal to v ever after P has passed through the ring; and this velocity can be measured by observing the time, t, taken to move over any measured height, h'; then $v=\frac{h'}{t}$, and if we put this value into (8), we obtain g.

As regards the magnitude of T, it is evident at first sight that it must be something between P and Q, the excess of P over T producing the downward acceleration of P, and the excess of T over Q the upward acceleration of Q.

What is the pressure on the pulley during the motion? The pulley is acted upon by the two downward tensions of the portions of the cord, whose resultant is $2T$—i.e. $\frac{4PQ}{P+Q}$; this is the pressure, and it is balanced by the wall, or other support, to which the pulley is fixed.

As an example of two bodies moving in contact, suppose a mass of weight P (fig. 60) placed on a body, AB, which is moved vertically in a given manner; what is the pressure between the two bodies?

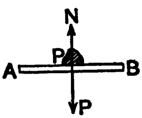


Fig. 60.

Suppose, for example, that AB is, by any means, moved downwards with a constant acceleration of $2\ {}^{f}/_{ss}$. Let N be the normal pressure between the bodies; then the resultant downward force acting on P is $P-N$; and, by hypothesis, the acceleration of P is $2\ {}^{f}/_{ss}$ downwards; hence (Art. 14),

$$\frac{P-N}{P}=\frac{2}{g}=\tfrac{1}{16};\ \therefore\ N=P(1-\tfrac{1}{16})=\tfrac{15}{16}P.$$

Next, let AB be moved upwards with an acceleration of $2\ {}^{f}/_{ss}$; then the resultant upward force on P is $N-P$; hence,

$$\frac{N-P}{P}=\frac{2}{g}=\frac{1}{16}, \therefore N=P(1+\tfrac{1}{16})=\tfrac{17}{16}P.$$

Next, let AB be moved with constant *velocity*; then, whether this velocity is upwards or downwards, the *acceleration* is zero,

$$\therefore N=P.$$

Next, let the system, P, AB, be simply allowed to fall freely; then the acceleration of each is g downwards, and the equation of P is

$$\frac{P-N}{P}=\frac{g}{g}=1, \therefore N=0.$$

There is no pressure between them in this case. If we put one coin on top of another and allow them to drop freely, there is no pressure between them.

Generally, if AB is moved downwards with an acceleration of $a\ {}^{f}/_{ss}$,

$$N=P\left(1-\frac{a}{g}\right);$$

and if it is moved upwards with a,

$$N=P\left(1+\frac{a}{g}\right).$$

If it is moved downwards with an acceleration $>g$, the pressure becomes negative—*i.e.* in order to keep the bodies in contact they must be held together with thread or cement, which will then be in a state of *tension.*

EXAMPLES

1. ACB (fig. 61) is a double-inclined plane; the inclination, i, of AC is $\tan^{-1}\frac{3}{4}$, and the inclination, i', of BC is $\tan^{-1}\frac{5}{12}$; a mass, P, of 40 pounds on AC, and a mass, Q, of 26 on BC are connected by a thin, flexible,

inextensible cord passing over a pulley fixed at C; the coefficient, μ, of friction between P and AC is $\frac{1}{4}$, and the coefficient, μ', of friction between Q and BC is $\frac{1}{6}$; find the motion of each body and the tension of the cord.

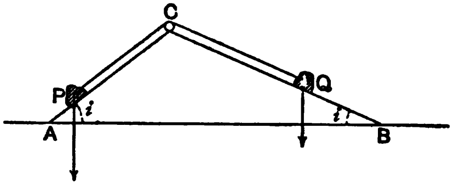


Fig. 61.

Since the cord is inextensible, the velocity and acceleration of Q upwards along BC are equal to the velocity and acceleration of P downwards along CA. Let the common acceleration be α, and let T be the tension of the cord. The normal pressure of the plane on P is $P\cos i$, or $\frac{4}{5}P$, and the force of friction on P is therefore $\frac{1}{5}P$ acting upwards; so that the whole force acting on P down the plane is $\frac{3}{5}P-\frac{1}{5}P-T$—that is, $\frac{2}{5}P-T$. Hence the equation of motion of P is—

$$\frac{\frac{2}{5}P-T}{P}=\frac{\alpha}{g}, \text{ or } \frac{16-T}{40}=\frac{\alpha}{g}.$$

Similarly, the equation of motion of Q is—

$$\frac{T-\frac{7}{13}Q}{Q}=\frac{\alpha}{g}, \text{ or } \frac{T-14}{26}=\frac{\alpha}{g}.$$

From these we have—

$$\alpha=\frac{g}{33}; \quad T=14\tfrac{26}{33}.$$

2. In the last, if the constants of the plane AC are (i, μ), and those of BC are (i', μ'), the weights being P and Q, find the acceleration of P downwards and the tension.

Result. $$\alpha=\frac{P(\sin i-\mu\cos i)-Q(\sin i'+\mu'\cos i')}{P+Q},$$

$$T=\frac{PQ}{P+Q}(\sin i-\mu\cos i+\sin i'+\mu'\cos i').$$

3. Show that the values of α and T in Atwood's machine follow from the results in the last example.

4. A mass of 20 pounds placed on a rough horizontal table is connected by a light inextensible cord passing over the edge of the table with a mass of 12 pounds hanging freely, the coefficient of friction between the table and the first mass being $\frac{1}{2}$; find the tension of the cord and the distance moved through in 2 seconds.

Result. The tension is $11\frac{1}{4}$ pounds' weight; distance $=4$ feet.

5. A mass, P, on a rough horizontal table ($\mu=\frac{1}{5}$) is connected by a thin flexible inextensible cord passing over a pulley at the edge with a mass, Q,

which hangs freely; find the ratio of Q to P so that the system may move over 5 feet in 10 seconds.

Result. $\dfrac{Q}{P} = \frac{65}{318}$.

6. A mass of 191 grammes on a rough horizontal table is connected with a freely-hanging mass of 49 grammes by a fine cord passing over a pulley at the edge, and it is found that if the system starts from rest it moves over 16 inches in 4 seconds; find the coefficient of friction.

Result. $\mu = \frac{1}{4}$.

7. A balloon contains a spring balance to which a mass of 1 pound is attached, and the indication of the spring is a tension of 17 ounces' weight; to what height has the balloon ascended from rest in 10 seconds?

Ans. 100 feet. (The balloon is moving upwards with an acceleration of 2 $^{f}/_{ss}$.)

8. In the last, if the balance indicates a tension of 14 ounces' weight, what is the nature of the motion of the balloon?

Ans. The balloon has a downward acceleration of 4 $^{f}/_{ss}$.

9. In Atwood's machine one mass, P, is 51 grammes, and the other, Q, is 49 grammes; when the system has moved through 8 feet from rest, 4 grammes are suddenly removed from P; how long and how far will the motion continue before reversal takes place?

Ans. The reversal will take place at the end of $4\frac{4}{5}$ seconds, and the distance = 7.68 feet.

(Observe that at first the downward acceleration of P is $\dfrac{g}{50}$, or $\frac{16}{25}$ $^{f}/_{ss}$; its velocity at the end of 8 feet is $\frac{16}{5}$ $^{f}/_{s}$; and when the 4 grammes are removed the acceleration of P becomes $\frac{2}{3}$ $^{f}/_{ss}$ upwards.)

10. To the ends of the cords in Atwood's machine are attached two scale pans, the mass of each of which is 4 ounces; in one is placed a mass of 12 ounces, and in the other a mass of 10 ounces; find the pressures between these masses and the pans in which they are placed.

Result. $11\frac{1}{5}$ and $10\frac{2}{3}$ ounces' weight.

(The acceleration is $\dfrac{12-10}{12+10+8}.g$, or $\dfrac{g}{15}$, so that if N is the pressure between the 12 oz. mass and its pan, the equation of motion of this mass is $\dfrac{12-N}{12} = \dfrac{a}{g} = \frac{1}{15}$; ∴ etc.)

11. To the ends of the cord in Atwood's machine are attached two equal scale-pans, each of mass p, and masses P and Q are placed in them; find the pressure between each mass and its pan.

Result. $\dfrac{2P(Q+p)}{P+Q+2p}$ and $\dfrac{2Q(P+p)}{P+Q+2p}$.

12. The masses P and Q in Atwood's machine are allowed to move from rest through any distance, x; if $P > Q$ what mass must be suddenly removed

from P at the end of the distance x so that the motion in the same sense shall continue through a further distance of $\frac{x}{n}$?

Ans. $$(n+1)\frac{P^2-Q^2}{(n+1)P-(n-1)Q}.$$

EXAMINATION ON CHAPTER VI

1. If a moving point has now a velocity of 10 $^f/_s$, and its velocity is uniformly increased at the rate of 2 feet per second per minute, what will be its velocity at the end of a quarter of an hour?

2. Give the expression connecting the velocity, acceleration, and time in the case of uniformly accelerated motion.

3. What is meant by a *velocity diagram*?

4. What is the curve which represents the velocities of a point at different times when the motion is uniformly accelerated?

Is the curve different if the velocity is uniformly retarded?

5. What, in this diagram, represents the distance travelled over in a given interval of time? Why does the area of the diagram represent this distance? (Because, however the velocity may vary, it can always be regarded as constant throughout an extremely small time.)

6. Give the expression for the *distance* in terms of the *time* in uniformly accelerated motion.

7. Give the relation between the *velocity* and the *distance* travelled in this case.

8. What is meant by the vertical height due to a given velocity? What height is due to a velocity of 2000 $^f/_s$? (62,500 feet.)

What is meant by the velocity due to a given height? What velocity is due to a height of 100 feet? (80 $^f/_s$.) Give a simple rule for calculating the height of a tower, or the depth of the surface of the water in a well, from the time taken by a stone to fall.

9. What is the object of Atwood's Machine?

10. State why it is clear without calculation that the tension of the cord in Atwood's Machine is not equal to—

(*a*) the difference of the weights,
(*b*) the sum of the weights,
(*c*) the greater weight.

11. If one penny is placed on top of another and both are then allowed to fall freely, what is the pressure between them?

12. If a man stands in a lift, what pressure does he exert on it when—

(*a*) the lift is moving upwards with constant speed?
(*b*) the lift is moving downwards with constant speed?

13. How can the lift move so that the pressure is—

(*a*) greater than his weight?
(*b*) less than his weight?

14. If you are in a balloon and have a mass of 1 pound suspended from a spring balance, and if the balloon indicates a pull of 17 ounces' weight, what do you infer?

CHAPTER VII

IMPULSE AND MOMENTUM; WORK AND ENERGY

36. **Impulse of a Force.**—If a force, P, of constant magnitude, acts on a particle for a time t the product

$$P.t$$

is called the *impulse of the force* for the time t.

Thus, if P is 4 pounds' weight and t is 2 seconds, the impulse is 8 "pounds' weight-seconds." The unit of impulse—which in this case would be the impulse of a force of 1 pound weight acting for 1 second—has not received any short name, so that we cannot speak of it otherwise than as above. If P is measured in poundals, $P.t$ is spoken of as "poundal-seconds"; and so on.

Momentum.—If a particle of mass m is moving with a velocity v the product

$$m.v$$

is called the *momentum* of the particle. Thus, if a particle whose mass is 3 ounces has a velocity of 20 $^f/_s$, its momentum is 60 "ounce-foot-seconds." The unit of momentum—in this case that possessed by a mass of 1 ounce moving with a velocity of 1 $^f/_s$—has not received any short name, so that momentum cannot be spoken of in more concise terms. If a mass of 5 grammes has a velocity of 20 $^c/_s$, its momentum is 100 gramme-centimètre-seconds; and so on.

Connection between Impulse and Momentum.—Let a force of P pounds' weight act on a mass of m pounds, and produce an acceleration of a $^f/_{ss}$; then (Art. 14)

$$P=\frac{ma}{g}. \qquad \qquad (1)$$

Let P act constantly in the same right line for t seconds, and multiply both sides of (1) by t; then $P.t=\dfrac{mat}{g}$. But if u

and v are the velocities of the particle at the beginning and end of t, both measured in the sense of a or P, we have $at = v - u$, by (1), p. 73. Hence

$$P.t = \frac{m(v-u)}{g} \quad . \quad . \quad . \quad (2)$$

Now $P.t$ is the impulse of the force for the time t; mv is the momentum of the particle at the end, and mu the momentum at the beginning, of t; so (2) asserts that—

The impulse of a force is equal to the momentum which it generates, divided by g.

If P had been taken in *absolute*, instead of *gravitation*, measure, (1) would be replaced by

$$P = ma \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and (2) would become

$$P.t = m(v-u), \quad . \quad . \quad . \quad (4)$$

—i.e. *the impulse of the force is equal to the momentum which it generates*; so that whether we divide the momentum generated by g or not depends on whether the force is in gravitation or in absolute measure.

The student will do well to remember that in (2) and (4) the velocities v, u are both measured in the sense of the force, and that therefore if we call the momentum at the beginning of the time t the *old momentum* and that at the end of t the *new* momentum,

the impulse of a force = the new momentum in the sense of the force − the old momentum in the sense of the force,

divided or not by g, according as the force is taken in gravitation or in absolute measure.

Equation (2), or (4), is called the *Equation of Impulse and Momentum.*

As a numerical illustration, suppose that a force, P, acts on a mass of 2 ounces (fig. 62) which is moving with a velocity of $30\,{}^{f}/_{s}$ in a sense opposite to that of P, and that in $\frac{1}{2}$ second the velocity of the particle is reversed and converted into $50\,{}^{f}/_{s}$; what is the magnitude of P?

Here the new momentum in the sense of P is 2×50 ounce-foot-sec. units, and the old momentum in the sense of P is -2×30; therefore, if P is measured in gravitation units—*i.e.* in ounces' weight,

$$P \times \tfrac{1}{2} = \frac{2(50+30)}{32}$$

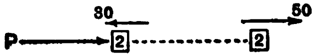


Fig. 62.

$\therefore\ P = 10$ ounces' weight.

If P is in absolute units, $P \times \frac{1}{2} = 2(50+30)$, $\therefore\ P = 320$; and the absolute of force here would be that which produces an acceleration of $1\ ^f/_{ss}$ in 1 ounce mass.

If the absolute unit of force here is a poundal, we must write the mass of the particle $\frac{1}{8}$ lb., and then $P \times \frac{1}{2} = \frac{1}{8}(50+30)$, $\therefore\ P = 20$ poundals.

If, when P began to act, the particle had a velocity of $30\ ^f/_s$ in the *same sense* as P, our equation of impulse and momentum would be—

$$P \times \tfrac{1}{2} = \frac{2(50-30)}{32}$$

$$\therefore\ P = \tfrac{5}{2} \text{ ounces' weight.}$$

Taking the data as in fig. 62, suppose that the change of velocity from $30\ ^f/_s$ to $50\ ^f/_s$ in the opposite sense is produced by P in $\frac{1}{640}$ second, what is the magnitude of P?

The equation of impulse and momentum is now—

$$P \times \tfrac{1}{640} = \frac{2(50+30)}{32}$$

$$\therefore\ P = 3200 \text{ ounces' weight.}$$

This shows that in order to change the velocity of a particle by any appreciable amount in an extremely short time, the applied force must be enormous—i.e. *enormous compared with the weight of the particle*; and the result is in accordance with common-sense, because the force with which the Earth pulls the particle (its weight) takes 1 second to produce a change of $32\ ^f/_s$ in the velocity of the particle; in such a time as $\frac{1}{640}$ second, for example, it would produce a change of only $\frac{32}{640}$, or $\frac{1}{20}$, feet per second; so that a force which in this time produces a very abrupt change of velocity must be vastly greater than the weight.

This prepares us for the consideration of the case in which a body receives a *Blow*—i.e. *the impulse of a force which acts on the body for some very small (and usually unknown) time, and yet produces a very abrupt change of velocity.*

Many simple instances of Blows may be cited. Thus, if we place a small piece of chalk on the table and flick it off with the finger, we apply to the particle for some small fraction of a second a force which is, perhaps, many hundreds of times its weight. Suppose that we suddenly send off the particle with a velocity of 40 $^f/_s$; then, if w is the weight of the particle, t the time of contact of the finger with it, and P the pressure of the finger against it, the equation of impulse and momentum is—

$$P.\,t = w\tfrac{40}{32} = \tfrac{5}{4}w.$$

Can we say what the magnitude of the force in this Blow is? No; not from the mere fact that the particle is sent off with 40 $^f/_s$: we cannot separate P from t in this equation, and in such cases it is usual to use B for the product $P.\,t$, and to write the equation of impulse and momentum—

$$B = \frac{w(v-u)}{g},$$

where B is called the *Blow* received by the particle. Observe, then, that this B is the *impulse of a force*—i.e. $P.\,t$, the product of a Force and a Time.

It is most important to remember that a *Blow* is not a *Force*: in every Blow there is *involved* a Force.

In the case of the piece of chalk flicked off the table, we cannot say what the magnitude of P is until we know *the time* for which it has acted. If it has acted for, say, $\frac{1}{128}$ second, we shall have—

$$P \times \tfrac{1}{128} = \tfrac{5}{4}w, \quad \therefore\ P = 160w.$$

A very common error is contained in such a question as this: A hammer strikes a mass of a pound, and sends it off with a velocity of 50 $^f/_s$; what is the force of the hammer?

It is quite impossible to answer this question: we know nothing about the *force* of the hammer—we know only the magnitude of the *blow* which it has given. Before the *force* exerted by the hammer can be known, the *time* during which it acts must be known.

EXAMPLES

1. A mass of 4 ounces strikes a fixed plane normally with a velocity of 60 $^f/_s$, and is reflected with a velocity of 36 $^f/_s$; what velocity would the blow which the body receives impart to a mass of 2 pounds?

If the force involved in the blow is measured in ounces' weight, we have, by the rule, p. 90,

$$P.\ t = B = 4\,\frac{[36-(-60)]}{32} = 12.$$

If B is applied to a mass of 2 lbs., we have $B = \frac{32v}{32} = v$, $\therefore v = 12\ ^f/_s$.

2. If the duration of the blow in the last is $\frac{1}{640}$ second, what is the force in the blow?

Ans. 7680 ounces' weight.

3. A mass of 5 kilogrammes strikes a fixed plane normally with a velocity of 1200 $^c/_s$, and is reflected with a velocity of 762 $^c/_s$, what velocity would the blow impart to a mass of 90 grammes?

Ans. 1090 mètres per second.

4. If the duration of contact with the plane in the last is $\frac{1}{10}$ second, what is the mean value, in kilogrammes' weight, of the pressure exerted on the plane?

Ans. 100.

5. A hammer whose mass is 2 pounds strikes a fixed steel plane with a velocity of 10 $^f/_s$; if the hammer rebounds after $\frac{1}{10000}$ second with a velocity of 6 $^f/_s$, what is the mean value of the pressure exerted on the plane?

Ans. 10,000 pounds' weight. (This shows how easy it is to indent a very hard plane by a blow if the *time* occupied by the blow is very small.)

37. **Work.**—If a force of constant magnitude and line of action acting on a body at a point A (fig. 63) displaces its point of application to any other position, B, along its line of action, the product

$$P \times AB$$

is called the *work done by the force* for this displacement.

Work is either positive or negative. When the displacement of the point of application of the force takes place in the sense in which the force acts—as from A to B in fig. 63—the work is positive. When the displacement takes place in the sense opposite to that of the force —as from A to B' in fig. 63—the work is negative. The

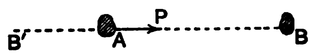


Fig. 63.

work done by P if its point of application is displaced from A to B' is

$$-P \times AB'.$$

[Of course, if nothing but the force P acts on the body, A would not be displaced to B'; for this displacement other forces than P must act. This, however, does not alter the fact that the work of P is $-P \times AB'$.]

If force is measured in pounds' weight and length in feet, work is foot-pounds' weight. Thus, if a force of 10 pounds' weight displaces its point of application through 4 feet in its own sense, the force does 40 foot-pounds' weight of work. If force is measured in poundals and length in feet, work is foot-poundals.

The unit of work has not received any short name, except in the C.G.S. system: when force is measured in *dynes* and length in *centimètres*, a unit of work is a dyne-centimètre, which is called by the special name—

erg.

Now, in fig. 63, if u is the velocity of the body in the position A, and v its velocity in the position B, we have, by (γ), p. 77,

$$v^2 - u^2 = 2a \times AB, \quad . \quad . \quad . \quad . \quad (5)$$

where a is the acceleration produced in the body by the force P. If w is the weight of the body, we know that

$$P = w\frac{a}{g}. \quad . \quad . \quad . \quad . \quad (6)$$

Multiply both sides of this equation by AB, and we have

$$P \times AB = \frac{w(v^2 - u^2)}{2g},$$

$$\text{or, } P.\,s = \frac{w(v^2 - u^2)}{2g}, \quad . \quad . \quad . \quad . \quad (7)$$

where s denotes AB.

Observe that, as in the case of impulse and momentum, p. 90, u and v are any two velocities of the particle, and s the distance between the corresponding positions of the particle.

Energy.—Energy means *capacity for doing work.* If a

particle of weight w has a velocity v, it can overcome any resistance, R, through a certain distance, and be brought to rest—*i.e.* it can do work by means of the velocity which it has. Let A (fig. 64) be the position of the particle when it has the velocity v; let it encounter a resistance of magnitude R at A, and drive the point of application of R to B, at which point its velocity is zero. Then, using equation (7), we have

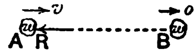


Fig. 64.

$$-R \times AB = \frac{w(0^2 - v^2)}{2g},$$

$$\text{or } AB = \frac{wv^2}{2g.R};$$

which shows that the distance through which the particle can overcome the resistance R when it yields up the whole of its velocity is obtained by *dividing* $\frac{wv^2}{2g}$ *by the resistance.*

The amount of work which the particle can do before it is brought to rest is $\frac{wv^2}{2g}$, and this is quite independent of the magnitude of the resistance against which it works.

A numerical example will tend to clearness. Let a particle whose mass is 2 ounces have a velocity of 40 $^f/_s$—

(*a*) What amount of work can it do before losing its velocity?

(*b*) Through what distance can it drive a steady resistance of 10 ounces' weight?

(*c*) What resistance will bring it to rest in 15 feet?

The amount of work that it can do is $\frac{2 \times 40^2}{64}$ foot-ounces' weight—*i.e.* 50 foot-ounces' weight.

If it works against a force of 10 ounces' weight and moves through s feet when its velocity comes to zero, $10 \times s = 50$, $\therefore s = 5$ feet.

If a force R ounces' weight stops it in 15 feet, $R \times 15 = 50$, $\therefore R = \frac{10}{3}$ ounces' weight.

If a particle of weight w having a velocity $v\ ^f/_s$, is set to work directly against its own weight—*i.e.* if the particle is

projected vertically upwards with the velocity v—how high will it go? By the above, if h is the height,

$$w \times h = \frac{w.v^2}{2g},$$

$$\therefore h = \frac{v^2}{2g},$$

which we found before (p. 78).

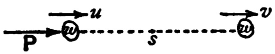
The energy, *i.e.* capacity for doing work, which a particle has in virtue of its velocity, is called its ***kinetic energy***.

If we imagine the particle to be placed at a height h above the ground, the weight, w, of the particle can do the amount of work

$$w . h$$

in moving the particle down to the ground. This capacity which the weight of the particle has for doing work in virtue of the *position* of the particle is generally called "potential energy"—an objectionable term which we shall not employ, although its use is almost universal. When we have occasion to speak of work which can be done by forces, or by particles, in virtue of the *positions* and not the *velocities* of the particles on which the forces act, we shall call such energy *static energy*.* This is not, however, a matter with which the beginner needs to trouble himself at present.

The equation (7), p. 94, which is for clearness represented in fig. 65, is called the ***Equation of Work and Energy***.



Fig, 65.

Fig. 65 represents a particle of weight w acted on by a constant force, P, having a velocity u in one position and a velocity v in any other position, the distance between the positions being s; then we have the result—

Kinetic energy of the particle in second position – kinetic energy in first = the work done on *the particle by the force from the first to the second position* . . (A)

* *Positional Energy* would also do; it is better than "Potential" Energy.

If the force acting on the particle is measured in *absolute* instead of *gravitation* units, of course (6), p. 94, becomes

$$P = wa,$$

and (7) becomes

$$P.s = \frac{w(v^2 - u^2)}{2},$$

and the kinetic energy of a particle of mass w pounds moving with a velocity v f/s is $\frac{w.v^2}{2}$ *foot-poundals.*

Thus, if a mass of 4 ounces has a velocity of 40 f/s, its kinetic energy is correctly described as $\frac{\frac{1}{4} \times 40^2}{64}$ foot-pounds' weight, or as $\frac{\frac{1}{4} \times 40^2}{2}$ foot-poundals; *i.e.* either $\frac{25}{4}$ or 200, according as we estimate it in foot-pounds' weight or in foot-poundals.

A simple diagram will assist the student to see the connection between the dynamical principles hitherto discussed.

$$P = w\frac{a}{g} \text{ (\textit{Newton's Axiom} II.)}$$

Impulse and Momentum | Work and Energy

$$P.t = \frac{w(v - u)}{g} \qquad\qquad P.s = \frac{w(v^2 - u^2)}{2g}$$

At the head stands Newton's Second Axiom—the foundation of Dynamics. If we multiply both sides of the equation expressing it by t, we obtain the equation of Impulse and Momentum; and if we multiply both sides by s, we obtain the equation of Work and Energy.

If in all the denominators we omit g, we obtain the same equations in the *absolute* system of measurement of forces.

Observe particularly that the kinetic energy of a particle being proportional to v^2, it will be the same whether v is + or − ; *i.e.* if a particle in any position has a velocity v it does

not matter whether the particle is moving towards the right or left or in any other direction at that point, its kinetic energy is the same—viz. $\dfrac{+wv^2}{2g}$.

EXAMPLES OF IMPULSE AND MOMENTUM, WORK AND ENERGY

1. A mass of 2 ounces moving with a velocity of 1280 $^f/_s$ encounters a steady resistance of 1600 pounds' weight; through what distance and for what time will the body move?

Express the kinetic energy of the body in foot-pounds' weight, and express the fact that it expends this in driving a resistance of 1600 pounds' weight through s feet; then

$$\frac{\frac{1}{8}\times 1280^2}{64}=1600\times s$$

$$\therefore\ s=2 \text{ feet.}$$

Express the momentum of the body in foot-second-pound units. It $=\frac{1}{8}\times 1280$; hence

$$\frac{\frac{1}{8}\times 1280}{32}=1600\times t$$

$$\therefore\ t=\tfrac{1}{320}\text{ second.}$$

2. If a force of 6 pounds' weight acts on a mass of 8 ounces through a distance of 9 inches, what velocity does it generate in the mass?

Ans. 24 $^f/_s$.

3. In the last, what is the time occupied by the force?

Ans. $\frac{1}{16}$ second.

4. If a mass of 20 kilogrammes is acted upon by a force of 1 million dynes through a distance of 1 mètre, what velocity wlll be acquired?

Ans. 1 mètre per second.

5. A train is moving with a velocity of 45 miles per hour; find the distance and the time in which it can be stopped by the brakes, supposing that their total pressure on the wheels amounts to $\frac{1}{8}$ of the weight of the train, and that the coefficient of friction between them and the wheels is $\frac{1}{8}$, neglecting all other resistances.

Result. 1089 yards; 99 seconds.

6. A train is moving with a velocity of 30 miles per hour; find the distance and the time in which it can be stopped by the brakes, supposing that their total pressure on the wheels amounts to $\frac{1}{4}$ of the weight of the

train, and that the coefficient of friction between them and the wheels is $\frac{1}{8}$, neglecting all other resistances.

Result. 605 feet; $27\frac{1}{2}$ seconds.

7. In the last, if, in addition to the brakes, there is a uniform resistance of 8 pounds' weight per ton, in what distance and in what time will the train be stopped?

Ans. $564\frac{4}{5}$ feet; $25\frac{3}{5}$ seconds.

[*The following examples may be omitted on first reading.*]

8. A railway train of weight W is to be brought from rest at a station, A, to rest at a station, B, a distance l, in t seconds, working all through against a uniform resistance, R, a certain constant pull being applied to the train for a portion of the time; calculate the magnitude of the pull and the time for which it must act.

Let P be the magnitude of the pull, which is applied to the train from A to some point, C, on AB, and then removed. Then, using the equation of work and energy from the first position, A, to the second position, B, since there is a zero velocity at each station, and since the work of P from A to B is $P \times AC$, and the work of R from A to B is $-R \times l$, we have, according to principle (A), p. 96,

$$0 - 0 = P \times AC - R.l,$$

$$\therefore \quad P.s = R.l, \qquad (1)$$

where $s = AC$. Again, if v is the velocity of the train at C (where the velocity is greatest), and t' is the time of the run from A to C under the total force $P - R$, we have from impulse and momentum—

$$(P - R)t' = \frac{Wv}{g}, \qquad (2)$$

and from the same principle for the run from C to B,

$$R(t - t') = \frac{Wv}{g}; \qquad (3)$$

hence from (2) and (3)

$$P.t' = R.t \qquad (4)$$

Now, if a is the acceleration of the train from A to C, we have $a = g\frac{P - R}{W}$; therefore

$$s = \tfrac{1}{2}gt'^2.\frac{P - R}{W}. \qquad (5)$$

But from (1) and (4) we have—

$$\frac{s}{t'^2} = \frac{Pl}{Rt^2}. \qquad (6)$$

Hence from (5) and (6)

$$\tfrac{1}{2}g\frac{P-R}{W}=\frac{Pl}{Rt'^2}$$

$$\therefore \quad P=\frac{R^2.gt^2}{R.gt^2-2Wl} \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

which determines P, while (1) and (4) give the values of AC and t'.

The least possible time in which the run from rest to rest could be accomplished is given by the denominator of the right hand side of (7); for, since P must be positive, we must have—

$$R.gt^2>2W.l,$$

$$\therefore \quad t>\sqrt{\frac{2W}{R}\cdot\frac{l}{g}},$$

the magnitude of R being, of course, supposed fixed.

9. Supposing that in the last question when the pull is removed, brakes are applied to the wheels, producing on each wheel a given normal pressure, N, and that the constant resistance, R, acts as before, calculate the necessary pull and the time of its action if the train is to be brought, as before, from rest to rest in t seconds.

If μ is the coefficient of friction between the brakes and the wheels, each wheel to which a brake is applied is acted upon by a frictional resistance equal to μN, and the work of this resistance from C to B is $-\mu N\times(l-s)$. If n wheels are supplied with brakes, the frictional work is $-n\mu N(l-s)$, or $-R'(l-s)$, if we put R' for $n\mu N$, the total amount of friction.

Our equations now corresponding to those in the last question are

$$0-0=(P-R)s-(R+R')(l-s);$$

$$\therefore (P+R')s=(R+R')l. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Also $(P-R)t'=W\frac{v}{g}$, and $(R+R')(t-t')=W\frac{v}{g}$; therefore

$$(P+R')t'=(R+R')t, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

while the acceleration, α, from A to C is still $\frac{P-R}{W}.g$, so that

$$s=\tfrac{1}{2}gt'^2.\frac{P-R}{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Now, from (1) and (2) we have

$$\frac{s}{t'^2}=\frac{P+R'}{R+R'}.\frac{l}{t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

therefore, from (3) and (4),

$$\tfrac{1}{2}g\frac{P-R}{W}=\frac{P+R'}{R+R'}.\frac{l}{t^2},$$

$$\therefore P=\frac{R(R+R')gt^2+2WR'l}{(R+R')gt^2-2Wl}, \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

while (1) and (2) determine s and t'.

The work done against the friction of the brakes is $R'(l-s)$, or

$$\frac{2WR'l^2}{(R+R')gt^2}.$$

10. A train of 100 tons is to be taken from rest to rest in $2\frac{1}{2}$ minutes over a distance of 3610 feet by a uniform pull against a constant resistance of 20 pounds' weight per ton; the brakes press the wheels with a total force of $\frac{2}{3}$ of the weight of the train, and the coefficient of friction between them and the wheels is ·18; find the magnitude of the pull and the time during which it must be applied.

Result. The pull is 1·98 tons' weight; time $=138\frac{1}{3}$ seconds.

11. Two trains are travelling on the same rails in the same sense, the one in front with a velocity of 30 miles per hour, and the one behind with 45 miles per hour, being separated by a certain distance when the guards see each other; the one in front is at once accelerated at the rate of $\frac{5}{16}$ feet per second per second, and the one behind retarded at the rate of $\frac{3}{8}$ feet per second per second. In this way a collision is *just* avoided: find the distance between them when the danger was perceived, the further distance travelled when they just begin to separate, and their common velocity at this point.

Result. 352 feet; 1568 feet; 54 $^f/_s$.

12. An engine is capable of exerting a pull of 3 tons' weight, and it is to carry a train from one station to the next, a mile apart, in 4 minutes; the constant rolling resistance is 20 pounds' weight per ton, and the brake resistance 400 pounds' weight per ton; find the greatest weight of train that can be so carried.

Result. $202\frac{26}{143}$ tons' weight.

13. The resistances being as in the last question, if the engine can exert a pull of P tons' weight, and the distance between the stations is m miles while the journey is to occupy n minutes, find the greatest weight of train.

Result. $$112P\frac{45n^2-22m}{45n^2+440m}.$$

[The equations, with the notation of example 9, are

$$(P+\tfrac{5}{28}W)s=\tfrac{3}{16}Wl,$$
$$(P+\tfrac{5}{28}W)t'=\tfrac{3}{16}Wt,$$

$$s=16\frac{P-\dfrac{W}{112}}{W}.t'^2.]$$

We have now to consider the work done by a force when its point of application is not displaced along the line of action of the force but in some inclined direction.

Let a force, P, act on a body at the point A (fig. 66), and suppose A to receive a small displacement, AA', to a neighbouring position, A'; then the work done by the force P—which force may be considered as constant in magnitude and direction during this displacement—is P multiplied by the projection, Am, of the displacement along the line of action of P. If from A' we drop the perpendicular $A'm$ on the line of action of P, Am is called the *orthogonal projection*, or simply the projection, of AA' along AB.

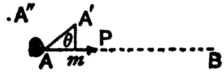


Fig. 66.

By definition, then, the work done by P in the displacement, AA', is

$$P \times Am.$$

This might be *negative*; for example, if the displacement of A takes place to A'' so that the foot of the perpendicular from A'' on AB falls to the left of A, the projection Am has a sense opposite to that of P, and the work is $-P \times Am$.

Another way of looking at the matter is this: the work of P for the displacement AA' is

$$P \times Am, \text{ or } P \times AA' \cos\theta, \text{ or } P\cos\theta \times AA',$$

where θ is angle between AA' and P's direction. Now, taking the last form, $P\cos\theta$ is the component of P along AA', and we can say that the work of P is the component of P along the displacement multiplied by the *whole* of the displacement; in other words, resolve P into components along and perpendicular to AA'; the work of the perpendicular component is zero, and the whole work is that due to the other component.

COROLLARY: *If the point of application of a force is displaced in a direction perpendicular to the force, the force does no work.*

A very simple and important case is that of the work done by gravity on a body which is displaced from any one position to any other. Suppose a body to occupy the position A (fig. 67), and to be displaced along any path whatever until it reaches the position B; let $g, g', g'' g''' \ldots$ be successive and very

close positions of the centre of gravity of the body—*i.e.* the point in the body at which its weight, W, acts. In the displacement from g to g' the work of W is $W \times gn$, if $g'n$ is drawn perpendicular to the vertical through g; gn is the *vertical* step from g to g'. Similarly, if $g'n'$ is the vertical step from g' to g'', the work done by W in the displacement from g' to g'' is $W \times g'n'$; and the next amount of work is $W \times g''n''$, and so on. Hence the total work done by W from A to B is

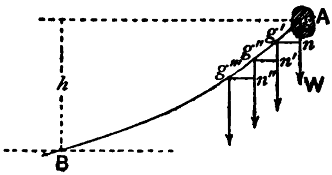


Fig. 67.

$$W(gn + g'n' + g''n'' + \ldots .),$$

or $W \times$ sum of all the *vertical* steps from A to B—*i.e.*

$$W . h,$$

where h is the *difference of level* between A and B.

If A and B are at the same height above a horizontal plane, the work done by W is zero, no matter what the nature of the path AB may be—it may have any number of humps and hollows.

If B is at a higher level than A, the work done by W from A to B is *negative.* No work is done by the weight of a particle if the particle is displaced in any way along a horizontal table.

We may also note that if any number of forces act at the same point, A, fig. 66, which is displaced to A', *the sum of the works done by the forces is equal to the work done by their resultant.* For, resolving all the forces along and perpendicular to AA', their total work is AA' multiplied by the sum of their components along AA'; and we know (Chap. III.) that the sum of these components is equal to the component of their resultant in the same direction.

38. General Equation of Work and Energy for a Particle.—We have already given the equation or principle of Work and Energy for a particle acted upon by a force constant in magnitude and line of action, the motion of the particle taking place in this line. We shall now prove that—*if a particle acted*

upon by any forces moves from any one position to any other, its kinetic energy in the second position, minus its kinetic energy in the first, is equal to the total work done on the particle by the forces from the first to the second position.

This is the general principle of Work and Energy for the motion of a particle.

Suppose that a particle describes any path, AB, fig. 68; let A_1 be any position of the particle, and R the *resultant* of all the forces acting on the particle in this position; let v_1 be the resultant velocity of the particle at A_1—*i.e.* the velocity along the tangent to the path at A_1; take on the path an indefinitely great number of close positions, A_1, A_2, A_3, . . . A_n, of the particle; let v_1, v_2, v_3, . . . v_n be the velocities of the particle in these positions.

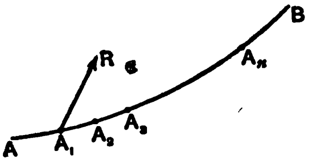


Fig. 68.

Resolve R into a component, S, along the tangent, A_1A_2, and a perpendicular component. Then if w is the mass of the particle, and we use gravitation measure of force, the acceleration a along A_1A_2 due to the force S is given by the equation

$$a = \frac{w}{S}.g.$$

Also by (γ), p. 77,

$$v_2^2 - v_1^2 = 2a.A_1A_2 = 2\frac{S.A_1A_2}{w}g$$

$$\therefore \frac{wv_2^2}{2g} - \frac{wv_1^2}{2g} = S.A_1A_2,$$

This equation asserts that the gain of kinetic energy in moving from A_1 to A_2 is equal to the work done on the particle by all the forces from A_1 to A_2.

Now, imagine the *direction* of v_2 at A_2 to be changed from A_1A_2 to A_2A_3 without any alteration of magnitude. This does not alter the kinetic energy of the particle, nor does it require the doing of any work; it is effected by the component of R, which is normal to the path.

Applying, again, the same equation to the motion from A_2 to A_3, we have—

$$\frac{wv_3^2}{2g} - \frac{wv_2^2}{2g} = \text{work done by forces from } A_2 \text{ to } A_3.$$

Similarly,

$$\frac{wv_4^2}{2g} - \frac{wv_3^2}{2g} = \quad ,, \quad ,, \quad ,, \quad A_3 \text{ to } A_4.$$

. . . .

Adding these results, we have—

$$\frac{wv_n^2}{2g} - \frac{wv_1^2}{2g} = \text{work done by forces from } A_1 \text{ to } A_n,$$

or if v is the velocity of the particle at A and v' its velocity at B,

$$\frac{wv'^2}{2g} - \frac{wv^2}{2g} = \text{total work done on particle from } A \text{ to } B. \quad (\alpha)$$

This is the general equation of Work and Energy for the motion of a particle.

Forces which do no work.—If a particle moves along any smooth curve or surface, the reaction of the curve or surface does no work, because the reaction is at every point perpendicular to the direction in which the particle is moving. (See Corollary, p. 102.)

Again, if the particle is attached to an inextensible cord which is always tight, the tension of the cord does no work, because it does not alter the length of the cord. This is obvious if one end of the cord is fixed, because the other end (to which the particle is attached) must be always moving at right angles to the tension; but it is true, even if the cord has no fixed end—if, for instance, the cord is attached to two moving particles—so long as the length of the cord is not altered by the tension. This latter result, however, we shall not for the present require.

Particle sliding down any smooth surface under action of gravity.—If a particle slides from rest down any smooth curve or surface under the action solely of its weight and the (normal) reaction of the surface from any point, A, to any other point, B (fig. 67), the velocity acquired is simply

$$\sqrt{2gh},$$

where h is the difference of level between A and B. For the

only work done on the particle from A to B is $w \times h$, and since kinetic energy at B is $\frac{wv^2}{2g}$, and kinetic energy at A is zero, we have—

$$\frac{wv^2}{2g} - 0 = w \times h, \quad \therefore \; v = \sqrt{2gh}.$$

In this case the particle would ascend the surface beyond B, and reach a point at the same level as A, then return to A; and so on.

If a particle is projected from a point, B, along any smooth surface with a velocity v, it will reach a point whose *vertical* height above B is $\frac{v^2}{2g}$—the same height as that which it would attain if it were projected vertically upwards from B.

But there arises naturally the question: What is the use of discussing the motion of a particle on a *smooth* curve, since no such thing as an absolutely smooth curve exists? Doubtless, no such thing exists; and yet a particle can be made to move down any assigned curve whatever, without the intervention of any friction. Consider, for example, a circle, OPA, fig. 69, in a vertical plane. One way of attempting to make a particle fall down a smooth vertical circle is to make such a figure out of a steel wire and allow the particle, A, to slide down along it. This is a very bad way, and, after a few oscillations to and fro about the lowest point, O, the particle would be brought to rest by friction.

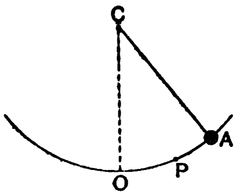


Fig. 69.

A practically perfect way of succeeding, however, is very easily found. Attach the particle to a very fine thread—a silk fibre, for example—and fix one end, C, of this thread. Now, if we allow the particle to fall from A, it will describe a circle of radius CA; and the only causes tending to stop the motion are the resistance of the air and a very small amount of friction, or perhaps resistance to bending, at C. These are much more feeble than the friction of a wire coinciding with the circle on which the particle could slide.

Such an arrangement is called a circular *pendulum*. Another

kind of pendulum has been used—viz. a *cycloidal pendulum.* In this case the particle, or bob, virtually slides down a smooth cycloid—not by making a cycloid of wire, but by attaching the particle to a fine thread, as above, fixing one end, *C*, then bending the thread over the surface of a cycloid, *CDE* (fig. 70), accurately cut out in brass or wood, and attaching the particle to the other end, *A*, of the thread. If *E* is the lowest point of the cycloid, *CDE*, and the length of the thread is equal to the length of the arc, *CDE*, when the particle is let go, the thread will *unwind itself* off the arc, *CDE*, and *A* will accurately describe a cycloid, *EAOE'*, exactly equal to that of which *CDE* is one-half. Of course, there will be an equal cycloidal rim, *CE'*, to the left, on which the thread will wrap itself as the particle moves towards the left past *O*. The tension of the thread takes the place of the normal reaction which would exist if the particle were actually sliding down the smooth wire *EOE'*; and the only causes tending to stop motion are air resistance and a very small resistance to bending in the thread. By properly choosing the form of the constraining curve or rim, and causing a thread to wind and unwind itself on and from it, the particle *A* may be made to fall down any assigned curve whatever; so that the case of its moving in this curve under gravity and without friction is realised. The curve formed by the constraining rim is the *evolute* of the curve which the particle is designed to describe.

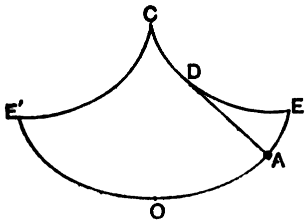


Fig. 70.

The following examples are meant to be solved by the direct application of the principle of work and energy. Of course, they can all be solved by using the equation expressing *Newton's Axiom* II. (see diagram, p. 97), from which the principle is derived.

EXAMPLES

1. A particle slides down a smooth inclined plane; what is its velocity at the foot of the plane?

Let *A* be the initial position of the particle (fig. 71). The forces acting on the particle are its weight, *W*, and *N*, the normal

reaction. If v is the velocity at the foot, B, the kinetic energy of the particle at B is $\frac{wv^2}{2g}$; the kinetic energy at A is zero; and the work done *by* the forces *on* the body from A to B is $w \times h$, where $h = AC =$ the height of the plane, the normal reaction, N, doing no work. (See *Cor.*, p. 102). Hence the equation of work and energy is—

$$\frac{wv^2}{2g} = wh$$

$$\therefore \quad v = \sqrt{2gh},$$

as, of course, we could also find by other elementary means.

2. AB (fig. 71) is a smooth plane inclined to the horizon at $\tan^{-1} \frac{3}{4}$, the length, AB, being 15 feet; a particle is projected at A with a velocity of 32 $^f/_s$ —

(1) downwards along the plane,
(2) upwards along the plane,

with what velocity will it reach B?

Let v be its velocity at B; then its kinetic energy at B is $\frac{wv^2}{2g}$, and its kinetic energy at A is $\frac{w \times 32^2}{2g}$, *whether it is projected downwards or upwards*; therefore in both cases the equation of work and energy is—

$$\frac{wv^2}{2g} - \frac{w \times 32^2}{2g} = w \times AC = w \times 9$$

$$\therefore \quad v^2 = 32^2 + 64 \times 9$$

$$\therefore \quad v = 40 \ ^f/_s.$$

The velocity with which it reaches B is the same in both cases. In case (2) the particle would travel up the plane beyond A, come to rest for an instant, and fall down, reaching B with a *downward* velocity of 32 $^f/_s$, because if it reaches a point, P, beyond A when it comes to rest, the difference of level of P and A being x feet, the work done on it by w from A to P is $-w.x$; and when it comes down from P to A, the work done on it in this motion is $+w.x$, so that the work done on it in the motion APA is zero, and therefore its kinetic energy on returning to A is the same as it was at first. (See remark on p. 98.)

3. If the plane AB in fig. 71 is rough, with a coefficient of friction equal to $\frac{1}{2}$, and inclination $\tan^{-1} \frac{3}{4}$, and a particle is allowed to slide down from A, what will be its velocity at B, AB being 15 feet?

Since the tangent of the inclination of the plane is greater than the coefficient of friction, the particle will slide down (Art. 32); the force of friction will therefore act up the plane and be $\frac{1}{2}N$, where N is the normal pressure. Now, there being no motion

perpendicular to the plane, $N = w\cos i = \frac{4}{5}w$, $\therefore$ the force of friction $= \frac{2}{5}w$. Use now the principle of work and energy. If the velocity of the particle at B is v, the gain of kinetic energy from A to B is $\frac{wv^2}{2g}$, there being no kinetic energy at A. There are three forces acting on the particle—viz. w, N, and $\frac{2}{5}w$; the work done by the first from A to B is $w \times AC$, or $9w$ foot-pounds' weight (if w is in pounds); the work done by the friction from A to B is $-\frac{2}{5}w \times AB$, or $-6w$; hence, by equation of work and energy,

$$\frac{wv^2}{2g} = 9w - 6w = 3w,$$

$$\therefore v = 8\sqrt{3}\ {}^{f}/_{s},\ g \text{ being taken as } 32\ {}^{f}/_{ss}.$$

4. If in the last the particle is projected upwards along the plane with a velocity $v\ {}^{f}/_{s}$, how far will it move up, and with what velocity will it reach A in the return motion?

Let it reach a point P, such that $AP = s$ feet, and stop for an instant. Then we may either put the case thus: the particle, in reaching P, expends its kinetic energy $\frac{wv^2}{2g}$ which it had at A in doing work against the resistances w and $\frac{2}{5}w$ (the latter being friction); or else make a formal use of the equation of work and energy, regarding P as the second position and A as the first. If we treat it in the first way we have,

$$\frac{wv^2}{2g} = w \times s\sin i + \tfrac{2}{5}w \times s, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$= ws(\tfrac{3}{5} + \tfrac{2}{5}) = ws;$$

$$\therefore s = \frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If we treat it in the second way, we have kinetic energy in second position $= 0$; kin. en. in first $= \frac{wv^2}{2g}$; total work done on particle from first to second position $= -w \times s\sin i - \frac{2}{5}w \times s$;

$$\therefore 0 - \frac{wv^2}{2g} = -w \times s\sin i - \tfrac{2}{5}w \times s,$$

which is identical with (a).

To find the velocity with which the particle reaches A from P, use the equation of work and energy from P to A. If v' is the velocity, since kin. en. at P is zero,

$$\frac{wv'^2}{2g} = w \times PA\sin i - \tfrac{2}{5}w \times PA = w.s(\tfrac{3}{5} - \tfrac{2}{5}) = \frac{w}{5} \cdot \frac{v^2}{2g}, \text{ by } (b);$$

$$\therefore v' = \frac{v}{\sqrt{5}}.$$

5. BA (fig. 71) is a rough inclined plane on which a particle is placed at A, the coefficient of friction being $\frac{1}{2}$; the distance BA is 10 feet, $\tan i = \frac{3}{4}$; the particle slides down to B and then along a rough horizontal plane, BQ, for which the coefficient of friction is $\frac{1}{3}$; how far will the particle move along this plane?

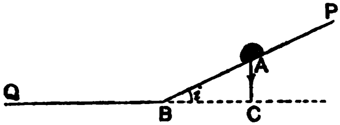


Fig. 71.

Use at once the equation of work and energy. Let the particle come to rest at Q. Then kinetic energy at $Q = 0$, kinetic energy at $A = 0$, $\therefore$ total work done on particle by all the forces from A to $Q = 0$. Now, while the particle is on the inclined plane, the forces which do work are w and the force of friction $\frac{2}{5}w$; and on the horizontal plane the only working force is the force of friction, $\frac{1}{3}w$. The total work done by w is $w \times BA \sin i$, or $6w$; the work done by the force of friction on the incline is $-\frac{2}{5}w \times BA$, or $-4w$; and the work done by the friction from B to Q is $-\frac{1}{3}w \times BQ$; therefore the equation of work and energy from A to Q is

$$0 - 0 = 6w - 4w - \tfrac{1}{3}w \, . \, BQ,$$
$$\therefore \; BQ = 6 \text{ feet.}$$

Note that there must not be a sharp edge of junction at B between the inclined and the horizontal plane; for if there were, the particle would receive a *blow*, and therefore abrupt change of velocity, from the horizontal plane. The junction must be rounded off. This will be better understood afterwards.

6. A particle slides down a rough inclined plane of inclination i and coefficient of friction μ; find its velocity at the base.

If the particle slides from rest at A (fig. 71), and $BA = l$, the velocity, v, at B is given by the equation—

$$v^2 = 2gl(\sin i - \mu \cos i).$$

7. A particle placed at a given point, A (fig. 71), on a rough plane of inclination, i, and coefficient of friction μ is projected up the plane with a given velocity, v; find the velocity with which it returns to A, and its velocity at the foot of the incline.

(Of course, the inclination of the plane must be greater than the angle of friction, otherwise the particle would never return to A.)

Result. If v' is the velocity with which the particle moves through A on the return journey,

$$v'^2 = \frac{\sin i - \mu \cos i}{\sin i + \mu \cos i} \, . \, v^2,$$

and if V is the velocity at B, and $BA = l$,

$$V^2 = \{v^2 + 2gl(\sin i + \mu \cos i)\} \, . \, \frac{\sin i - \mu \cos i}{\sin i + \mu \cos i}.$$

8. A train moving with a speed of 30 miles per hour comes to the foot of an incline of 1 in 112, and shuts off steam; the resistance of the road is 10 pounds' weight per ton; how far will the train run up the incline before stopping?

Ans. $2258\frac{2}{3}$ feet. (The *sine* of the inclination is $\frac{1}{112}$.)

9. A train, with steam shut off, runs a mile down an incline of 1 in 112, and with the velocity acquired runs up an incline of 1 in 56 through a distance of $\frac{1}{8}$ mile; what is the resistance to motion in pounds' weight per ton?

Ans. 10.

10. A train, with steam shut off, runs with a speed of 30 miles per hour down an incline of 1 in 120; find the resistance, in pounds' weight per ton, necessary to stop it in half-a-mile.

Result. $44\frac{1}{3}$.

11. If in a locomotive engine D is the diameter of the driving wheels, d the diameter of each cylinder, l the length of the stroke of the piston, and p pounds' weight per unit area the intensity of pressure of the steam, prove that the pull exerted by the engine on the train is

$$\frac{pld^2}{D}.$$

(The distance travelled by the engine during one stroke of the piston is $\frac{1}{2}\pi D$; therefore if P is the pull exerted on the engine by the rest of the train, the work of this force on the engine is $-\frac{1}{2}\pi D \times P$. The force exerted by the steam on each piston is $\frac{1}{4}\pi d^2 p$, $\therefore$ the work done by the steam on the engine in a stroke is $\frac{1}{2}\pi d^2 p \times l$. If there is no gain of kinetic energy of the engine in this motion, the total work done on the engine $=0$; $\therefore$ $\frac{1}{2}\pi d^2 pl - \frac{1}{2}\pi DP = 0$, and P is as given above. If p is estimated per square inch, l, d, D must all be taken in inches.)

The details of the driving of a railway train by steam, as involved in Example 11, lead to the consideration of the *transformation of energy*. Thus, let us begin with the contemplation of a mass of coal in the furnace. The *static energy* of this coal is the cause from which all the motion arises. When the coal is ignited, this static energy passes into the kinetic form called *heat*; this heat is taken up by the water in the boiler—still as kinetic energy, until the water begins to boil, and then commences a process in which the kinetic energy (or a part of it) is used up in separating the water molecules from each other, overcoming their mutual attractions, and converting the water into steam of high pressure. This pressure now does work on the piston, and a part of this work is at first employed in increasing the kinetic energy of motion of the whole train, while the rest of it is continuously absorbed in overcoming external resistances to motion. This is not the end of the process; for the kinetic energy of the train is itself being continuously converted into heat in the rails and in the atmosphere. Thus, by several stages, the original static energy of the coal finally passes into the

kinetic energy of heat which is radiated away into space. Only a very small fraction (about $\frac{1}{10}$) of the static energy of the coal is, in the process, capable of being used as work in driving the piston: hence the process employed in a locomotive engine is an extremely wasteful one.

An animal working any kind of machine—a man driving a bicycle up an incline, a horse dragging a boat against a stream, for example—is exactly comparable with a steam engine in its transformations of energy, the food which the animal consumes answering to the coal supply of the engine.

A curious consideration in connection with *animal locomotion* presents itself to us here.

No body can move with the slightest acceleration unless it is acted upon by some force in the direction and sense of the acceleration. This is Newton's Second Axiom. Suppose a horse dragging a boat through a canal by means of an attached rope. Though the horse may appear to move uniformly—*i.e.* with a speed which never varies during even the millionth of a second, and with a motion exactly like that of a smooth glass marble moving in a right line along a perfectly smooth horizontal glass plate—such is not the case in reality. The horse goes by jerks; and during the effort of each jerk his velocity varies considerably; that is, there is acceleration during the time of the effort made by his hoof with the ground. What force is there to account for this forward motion of the horse? Certainly not the tension of the rope, which acts on him in the wrong sense; nor the resistance of the air, for the same reason. Nothing remains but the tangential reaction of the ground—a kind of stress between his hoof and the ground, which we call the *force of friction*.

It sounds very paradoxical to say that the only force which produces the forward motion of the horse is the force of friction—yet such is the case. If the velocity of the horse can be regarded as constant (without looking minutely into the state of affairs taking place during the effort of the hoof on the ground), the mean value of this force of friction (or tangential stress) is equal to the tension of the rope; but, in reality, this force of friction varies also by jerks: it must for some time during the hoof effort exceed the tension, otherwise the forward motion of the horse could not take place.

Work and Energy in Atwood's Machine.—The fact that the motion of two particles connected, as in fig. 59, p. 82, is one of uniform acceleration, is very readily seen by the principle of work and energy; and as, in the more advanced portion of our subject, we shall frequently use this principle in cases which are essentially the same as that of the simple Atwood's machine, we shall deduce the value of the acceleration, a, given in equation (3), p. 83, by work and energy.

Suppose that A and B (fig. 59, p. 82) are the initial positions of P and Q, and that $AP=BQ=x$. Now, in the motion from A to P the work done on P is $P \times x +$ the work of the tension. Denote the work of the tension from A to P

by $-W$. (Of course, if we know that the tension is constant, its work on P is $-T\times x$; but we need not assume that T is the same throughout the motion.) Also, the tension does on Q the work $+W$ from B to Q, whereas gravity does on Q the work $-Q\times x$.

And if v is the velocity of P in the second position, the equation of work and energy for P from A to P is—

$$P\frac{v^2}{2g}=Px-W, \quad . \quad . \quad . \quad . \quad (1)$$

and the equation for Q is—

$$Q\frac{v^2}{2g}=-Qx+W, \quad . \quad . \quad . \quad (2)$$

since the velocity of Q is always equal to that of P. By addition the work of the tension disappears, and we have—

$$(P+Q)\frac{v^2}{2g}=(P-Q)x, \quad . \quad . \quad (3)$$

$$\therefore \quad v^2=2\frac{P-Q}{P+Q}g\,.\,x \quad . \quad . \quad (4)$$

Now, whenever v^2 is proportional to x, the distance moved over, the motion is uniformly accelerated (see art. 34), and if a is the acceleration,

$$v^2=2a\,.\,x.$$

Comparing this with (4), we see that

$$a=\frac{P-Q}{P+Q}\cdot g,$$

as before obtained. We have now shown that the motion is uniformly accelerated; and from this it follows that the tension is constant; for, since the motion of P is uniformly accelerated, the resultant force, $P-T$, acting on P must be constant, and its value is obtained from (1) by putting $W=T.x$ and using (4).

In the same way, the fact that the motion is uniformly accelerated can be deduced from the principle of *impulse and momentum*. For if I is the impulse of the tension in the time

t of moving from A to P, we have for the motions of P and Q, respectively,

$$\frac{Pv}{g} = P.t - I,$$

$$\frac{Qv}{g} = I - Q.t,$$

$$\therefore \quad v = \frac{P-Q}{P+Q}.gt,$$

which shows that v is uniformly accelerated, the acceleration being $\frac{P-Q}{P+Q}.g$.

Observe that since the tension is the same for P as for Q at each instant, its impulse has the same value for both, whether the tension is the same at all times or not. Of course if we knew that T is a constant, its impulse, I, could have been put equal to $T.t$.

EXAMPLE

A mass of weight W is placed at the foot of a rough inclined plane of inclination i and length l, and is connected by a thin flexible cord which passes over a pulley at the top of the plane with a mass of weight P which hangs freely; find the distance through which P must be allowed to fall so that if P is then detached, W may just reach the top.

Let v be the common velocity of P and W when P has descended a distance x; then, by work and energy,

$$(P+W)\frac{v^2}{2g} = \{P - W(\sin i + \mu \cos i)\}x \quad . \quad . \quad (1)$$

If P is now detached and W comes to rest at the top, the kinetic energy $W\frac{v^2}{2g}$ is expended in doing work against the resistance $W(\sin i + \mu \cos i)$, though the distance $(l-x)$; hence

$$W\frac{v^2}{2g} = W(\sin i + \mu \cos i)(l-x) \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we have, by division,

$$x = \frac{P+W}{P} \cdot \frac{\sin i + \mu \cos i}{1 + \sin i + \mu \cos i} \cdot l.$$

Power.—If a force P applied to a body at a point A (fig. 63, p. 93) displaces its point of application along its line of action and in its own sense, and if the velocity of the point in this direction is v, the product

$$P.v$$

is the *power* exerted at the instant by the force. This expression is evidently the time-rate at which the force is working. Thus, if P is 10 pounds' weight, and if A is displaced through 1 inch in $\frac{1}{60}$th part of a second, the work done by the force in this time is

$$10 \times \tfrac{1}{12} \text{ foot-pounds' weight;}$$

and as the force takes $\frac{1}{60}$ sec. to do this work, it is working at the rate of

$$10 \times \tfrac{1}{12} \times 60 \text{ foot-pounds' weight per second,}$$

—*i.e.* at the rate of 50 foot-pounds' weight per second. This time-rate of working is sometimes called the *activity* of the force.

Power, then, means *time-rate of doing work.* In this sense it is now used in all branches of applied science.*

The rate at which a strong horse can work is supposed to be represented by 550 foot-pounds' weight per second, or 33,000 foot-pounds' weight per minute; and this particular rate of working is called a *horse-power*, which is the unit of power in common engineering use. Another unit, called the *watt*, which is about $\frac{1}{746}$ of a horse-power, is a unit very commonly employed by electrical engineers.

When an agent has done 33,000 foot-pounds' weight of work, we must not conclude that he has worked with a horse-power. If he has done it in 1 minute, he *has* worked with 1 H.P. (the notation for a horse-power); but if he has taken a week to do it, his time-rate of working is a very small fraction of 1 H.P.

If, as in fig. 66, p. 102, the motion of the point, A, of application of a force, P, does not take place along the line

* Treatises on "Mechanics," some twenty or thirty years ago, invariably spoke of the "Power" and the "Weight" in various machines (such as the screw-press, the lever, the wheel and the axle, etc.) to designate what are now known as the "Effort" and the "Resistance." Old terms, however absurd, die hard; and hence, even in a few modern works, we still and the terms "power" and "weight" applied in this objectionable way.

of action of the force, the activity, or time-rate of working, of the force is the product of P and the *component* of the velocity of A along P's line of action. If the point of application is moving at right angles to the force, the activity of the force is zero.

EXAMPLES

1. An engine of 400 horse-power can draw a train of 200 tons up an incline of 1 in 280 at 30 miles an hour; find the resistance of the road in pounds' weight per ton.

Let p lbs.' weight per ton be the resistance; then the total resistance to this train is $200p$ lbs.' weight. Also, since the train is going with constant velocity, the positive work done on it by the steam from any one position to any other is equal numerically to the sum of the negative works done on it by the resistance of the road and by its weight; hence the time-rate at which the steam works is equal to the sum of the time-rates at which work is done against resistance and gravity. Now, $30\,^m/_h = 44\,^f/_s$, and the rate of doing work against the resistance of the road is

$$200p \times 44 \text{ foot-lbs.' weight per second.}$$

Also the vertical component of the velocity is $44 \times \frac{1}{280}\,^f/_s$, and the weight is 200×2240 lbs.' weight; therefore the rate of doing work against gravity is

$$44 \times \tfrac{1}{280} \times 200 \times 2240 \text{ ft.-lbs.' weight per sec.}$$

Hence the total time-rate of doing work against resistance and gravity is

$$200 \times 44(p+8),$$

which is $\dfrac{200 \times 44}{550}(p+8)$ horse-power. Equating this to the given horse-power, 400, we have

$$p = 17 \text{ lbs.' weight per ton.}$$

2. If an engine of horse-power H draws a train of W tons up a plane of inclination i with a uniform velocity of $v\,^f/_s$ against a resistance of p lbs.' weight per ton, prove that

$$H = \frac{W.v}{550}(p + 2240 \sin i).$$

3. Find the horse-power of an engine which draws a train of 120 tons with a velocity of 30 miles per hour up an incline of 1 in 224 against a resistance of 25 pounds' weight per ton.

Result, 336.

4. If in exercise 2 the engine draws the train *down* the plane, prove that

$$H = \frac{W.v}{550}(p - 2240 \sin i).$$

(Here gravity is working with, instead of against, the engine.)

5. Find the whole resistance to the motion of a steamer of 10,000 h.-p. which is being driven through the water with a constant speed of 20 miles per hour.

Result. $81\frac{118}{176}$ tons' weight.

6. The mass of a bicycle and its rider is 200 pounds; the machine and rider come down an incline of 1 in 100 with a uniform speed of 8 miles per hour against the resistance of the air, the pedals not being worked; prove that to go up an incline of 1 in 200 at the same speed the rider must work with ·064 of a horse-power.

If the radius of each crank is 6 inches, and the cranks make one revolution per second, show that the mean value of the crank effort (pressure on the pedal tangential to the circle) is about 11·2 pounds' weight.

7. The mass of the bicycle and rider being W pounds, if down an incline (small) of 1 in n there is a uniform speed of v feet per second, prove that to go up an incline of 1 in m at the same speed, the rider must work with $W\left(\frac{1}{n}+\frac{1}{m}\right)\frac{v}{550}$ horse-power.

8. Prove that when a bicycle and rider are coming down an incline against air resistance with uniform velocity there is no force of friction exerted on the wheels by the plane.

9. If a mass of 10 pounds is let fall from the top of a house, what is the activity of its weight when 25 feet have been fallen through?

Ans. 400 foot-pounds' weight per second.

EXAMINATION ON CHAPTER VII

1. Define the *impulse of a force*, and describe it in any system of units. Define the *momentum* of a particle, and describe it in any system of units.

2. Give the equation of impulse and momentum for a particle acted upon by a force which is constant in all respects for a time t.

3. If a force P acting on a mass w for a time t generates velocity v, are the equations

$$P.t = w.v \text{ and } P.t = w\frac{v}{g}$$

both true? If so, with what understanding?

4. If a hammer strikes a given particle and sends it off with a given velocity, can we speak of "the force of the hammer"? If so, can we say from these data what its magnitude is?

5. Define a *blow*. Why does a hammer blow indent a surface of steel, although the steel requires a steady pressure of very great magnitude to indent it?

6. Define *work*. When does a force do positive work and when negative?

7. What is the C.G.S. unit of work?

8. Define *energy*. Define *kinetic energy* and *static energy* ("potential" energy).

9. If the speed of one particle is twice the speed of another, how much more mass must the second have so that it may have the same kinetic energy as the first?

10. State the principle of work and energy for a moving particle, and give the equation expressing it.

11. Kinetic energy of a particle may be expressed either as $\frac{wv^2}{2}$, or $\frac{wv^2}{2g}$: with what understanding?

12. How are the equations of *impulse and momentum* and of *work and energy* deduced at once from "Newton's Second Axiom"?

13. When the displacement of the point of application of a force does not take place along the line of action of the force, how is the work done by the force from one position to another calculated? How represented in a diagram?

14. If at any instant a force is doing no work on a body, how must the point of application of the force be moving?

15. If a particle is lowered off a table to the floor and again raised to the height of the table, what is the work done on the particle by its weight?

16. Name any forces which in the motion of a particle do no work.

17. How may a particle be made to move on a *perfectly smooth* vertical circle? How on a *perfectly smooth* vertical cycloid?

18. Give the equation of work and energy for the Atwood machine motion.

How does this equation at once show that the motion is uniformly accelerated?

19. What is meant by *power*? What is the ordinary English engineering unit of power?

If the mass of a snail is ½ ounce and he climbs to the top of a house 40 feet high, what work has he done? Can you give the horse-power with which he has worked?

20. Define the activity of a force.

21. In the motion of a train driven by a steam engine state the various transformations of energy that occur.

CHAPTER VIII

BLOWS: COLLISIONS

In the previous chapter we have defined and illustrated the impulses of forces in various cases, the duration of the action of the forces being either short or long. In the present we shall consider more especially cases in which the duration is extremely small—impulses in such cases being generally termed blows.

39. **Impulse between two Bodies.**—Suppose two spheres of masses w and w' (fig. 72), to be moving with velocities v, v' in the line joining their centres, and to come into collision. They will remain in contact for, perhaps, some small fraction of a second, during which time their surfaces will be distorted, as represented in (*b*), and each will be subject at a given instant to a pressure, P, from the other. This pressure is a force which grows from zero up to a maximum; it is, therefore, different at different instants during the contact, and we have no ready means of figuring out its various values during this period. The compression of the two bodies will proceed for a certain time; it will then cease, and one or both

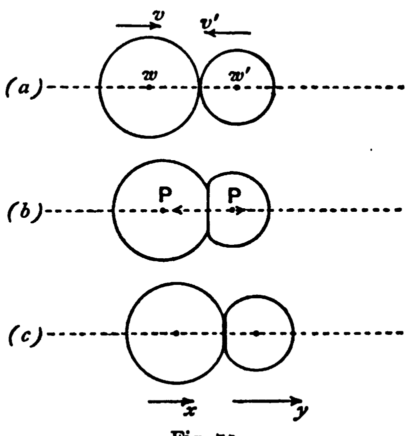


Fig. 72.

of the bodies will begin to regain its original figure, the pressure between them still, of course, continuing. Finally, the bodies will separate, as represented in (*c*), going with different velocities x, y, and at the instant of separation when, of course, P has fallen to zero, each will have partially regained its original shape — the extent to which the recovery of shape takes place depends on the nature of the bodies: two glass balls or two ivory ones would almost completely recover.

The most important things to note with regard to the pressure P are—

(1) however variable it may be during the impact, it is of the *same magnitude* for both bodies and in opposite senses;

(2) it acts for the *same time* on both.

If we imagine a diagram representing the various values of P during the time of collision, it must be something like fig. 73, in which OT is an axis along which *time* is represented, and Op an axis along which *force* is represented. If the whole time of contact is represented by $O\tau$, the curve whose ordinates represent the values of P, must shoot upwards from O with very great rapidity, and, after the maximum ordinate is attained, it must again drop down towards OT, crossing it at the point τ. The whole impulse of the pressure during the time $O\tau$ is represented by the area, $OPQ\tau$, of this curve, as is obvious from the perfectly analogous case of the distance travelled with variable speed, explained in fig. 55, p. 76.

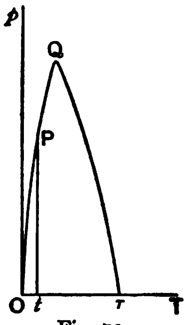


Fig. 73.

Hence, then, from (1) and (2) we conclude that

total impulse received by w = total impulse received by w',

these impulses being, of course, in opposite senses.

The *mean* value of P during the time, τ, of collision is, of course, such that

$$P \times \tau = \text{area of curve } OQ\tau.$$

As the figure $OQ\tau$ cannot be very different from a triangle,

we can see that the mean value of the pressure must be something like *half* the maximum value of the pressure.

If we denote the total impulse of P by B, we have, by the rule given in p. 90 for the motion of w,

$$B=\frac{w(-x+v)}{g}, \qquad (1)$$

the value of the force in the blow being taken in gravitation units. Similarly for the motion of w',

$$B=\frac{w'(y+v')}{g}; \qquad (2)$$

hence

$$wx+w'y=wv-w'v', \qquad (3)$$

which expresses the fact that the sum of the momenta in the line of collision after impact is equal to the sum of the momenta in the same sense before impact.

Our figure represents w and w' moving in opposite senses, so that the algebraic sum of their momenta from left to right before collision is $wv-w'v'$. The whole discussion would remain exactly the same if the two velocities v, v' were in the *same sense*, and then our momentum equation would be

$$wx+w'y=wv+w'v'. \qquad (4)$$

The fact that the impulses on the bodies are equal and opposite shows, then, at once that—*the resultant momentum of the system remains the same just after collision as it was just before*—all the momenta (both before and after) being measured in the *same sense*.

This important fact is called the principle of *the conservation of momentum*. We shall enlarge on this principle afterwards: it is one of great utility, and always readily applicable without any practical limitations such as those which hamper the use of the principle of work and energy.

We cannot determine the two unknowns x and y from the single equation (3): another relation between them is required, but it cannot be deduced by any dynamical considerations; it is to be found solely by experiment.

For the present we shall assume the bodies to be such that they do not separate at the end of the blow, but travel together with a common velocity.

In this simple case (approximately that of two lumps of putty) $x=y$, and the conservation of momentum gives at once

$$(w+w')x=wv\pm w'v', \quad . \quad . \quad . \qquad (5)$$

the $\pm$ depending on whether the original velocities are in the *same* or in *opposite* senses.

EXAMPLES

1. Two spheres whose masses are 6 ounces and 4 ounces collide in the line joining centres with velocities of 25 f/s and 35 f/s, respectively, in the same sense; if they do not separate, what is the common velocity after collision?

Ans. 29 f/s.

2. If they are moving in opposite senses before collision, what is the common velocity?

Ans. 1 f/s in the sense of the motion of the first sphere.

3. If in Ex. 1 the collision ends after $\frac{1}{64}$ second, what is the value of the mean pressure between the spheres?

Ans. 48 ounces' weight.

4. If in Ex. 2 the time of collision is $\frac{1}{64}$ second, what is the mean pressure?

Ans. 288 ounces' weight.

5. Masses of 4, 5, 8, 3 ounces are placed in a right line, a velocity of 30 f/s is communicated to the first; on collision no two of the masses separate; what is the final velocity of the system?

Ans. 6 f/s.

6. What are the magnitudes of the blows in the collisions which take place in Ex. 5?

Ans. $\frac{25}{12}$, $\frac{30}{17}$, and $\frac{9}{16}$ ounce-foot-sec. units.

7. A particle, A, hangs at rest from a fixed point, C, by means of a thread; a particle, B, of equal mass is attached to one end of a thread whose length is equal to that of the first, and whose other end is also attached to C; the line CB is drawn out from the vertical CA until B is at a height h above A; B is then let go and the particles do not separate on collision; to what height above A will they rise?

Ans. $\frac{1}{4}h$.

In all these cases there is a loss of kinetic energy in the system. Taking the general case in which a mass w moving with a velocity v overtakes w' moving with v' in the same sense—the case represented in fig. 72 except that v' is in the

opposite sense—and assuming that no separation takes place after collision, the common velocity, x, is given by the equation

$$x = \frac{wv + w'v'}{w + w'},$$

and the kinetic energy of the system after collision is $(w+w')\frac{x^2}{2g}$, which can be easily shown to be

$$\frac{wv^2}{2g} + \frac{w'v'^2}{2g} - \frac{ww'}{w+w'} \cdot \frac{(v-v')^2}{2g}.$$

The sum of the kinetic energies before collision is expressed by the first two terms, and the last term gives the loss of kinetic energy produced by collision.

We shall return to this farther on.

40. **Pile Driving.**—Suppose a pile of weight P (fig. 74) forced into the ground (bed of a river or other) by successive blows of a hammer, or of a very heavy body, W, which falls through a given vertical height h, is raised and again falls through h; and so on.

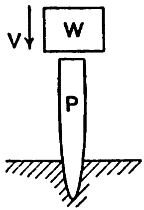


Fig. 74.

We shall assume two things—

(*a*) that the ground into which the pile is driven consists of merely loose substance, such as clay, mud, sand, or gravel.

(*b*) that on the occurrence of each blow the pile and the hammer travel downwards together with a common velocity—*i.e.* they do not separate.

The result of (*a*) is that only a finite force, comparable with the weight of the system, is produced by the ground on the pile from the very instant of collision; and such a force cannot in a small fraction of a second produce any appreciable momentum: it cannot, then, produce any effect *during the time of the blow*—that is, *the time which elapses between the instant of contact of hammer and pile and the instant at which they are travelling together with a common velocity.* (This time

is not, of course, the same as the time taken by hammer and pile to penetrate the ground and be brought to rest.)

We assume, then, that the ground is such that there is no blow at the foot of the pile.

If v is the velocity with which hammer and pile go together, and B is the upward blow received by W and the downward blow received by P, we have, by Art. 34,

$$B = \frac{W(V-v)}{g} \text{ for hammer, } \quad . \quad . \quad . \quad (1)$$

$$B = \frac{Pv}{g} \qquad \text{for pile, } \quad . \quad . \quad . \quad . \quad (2)$$

the force which involved in B being taken in gravitation units, as explained at p. 90. From these we have

$$(W+P)v = WV, \quad . \quad . \quad . \quad . \quad (3)$$

which asserts that the momentum of the system at the end of the blow is equal to that before the blow, a result which is obvious from the discussion in the last article.

The two bodies now go together with a common velocity, and the kinetic energy of the system at the end of the blow is $(W+P)\frac{v^2}{2g}$, i.e.

$$\frac{W^2V^2}{2g(W+P)}, \text{ or } \frac{W^2h}{W+P}. \quad . \quad . \quad . \quad . \quad (4)$$

Observe also the loss of kinetic energy to the system here. At the instant just before the blow the kinetic energy of the system was wholly in the hammer and was Wh; so that an amount of energy equal to

$$Wh - \frac{W^2h}{W+P}, \text{ or } \frac{WPh}{W+P} \quad . \quad . \quad . \quad (5)$$

has been lost to the system. (We shall so far anticipate future discussion as to say that the amount of kinetic energy which always disappears from a system of two bodies in collision is converted into the kinetic energies of the molecules of the bodies and of the surrounding air—rapid vibratory motions, producing sound and heat. Instead of speaking, as here and at the end of last article, of *lost* kinetic energy, it would

be more accurate to speak of the amount of kinetic energy *transformed* by the blow.)

Now (4) is the amount of kinetic energy with which the system starts to do work; and what is the resistance against which this work is done? If R is the resistance of the ground, the total force against which the hammer and pile have to work is

$$R-W-P,$$

since $W+P$ acts downwards on the system. If, then, the point of the pile is driven a distance x into the ground by one blow, the principle of work and energy gives

$$\frac{W^2h}{W+P}=(R-W-P)x, \quad . \quad . \quad . \quad (6)$$

supposing the resistance R to be constant throughout the penetration.

To find the time occupied in this penetration, use the equation of *impulse and momentum*. If t is the time in which a mass $W+P$ moving with a velocity v is stopped by a force $R-W-P$,

$$(R-W-P)t=\frac{(W+P)v}{g}=\frac{WV}{g}.$$

Hence substituting for $R-W-P$ from (6),

$$t=\frac{W+P}{W}\cdot\frac{2x}{V}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we measure the amount by which the pile is crushed by each blow, we can calculate the pressure between the hammer and pile during the blow, supposing it uniform.

For, the kinetic energy transformed in the blow has passed into the vibratory form by means of the crushing of the pile. If S is the mean pressure between hammer and pile during the blow, and z is the amount of compression of the pile, the energy expended must be $S.z$. Equating this to the energy transformed (5),

$$S=\frac{WP}{W+P}\cdot\frac{h}{z}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The *time* occupied by the blow is found by the principle

of impulse and momentum. If τ is the time, $S\times\tau=$ the momentum of the hammer at the beginning—its momentum at the end of the blow; and thus

$$S.\tau=\frac{W(V-v)}{g}=\frac{WP}{W+P}\cdot\frac{V}{g},$$

$$\therefore\ \tau=\frac{2z}{V}. \qquad (9)$$

The case in which a hammer drives a nail into a board is exactly the same in principle as this.

EXAMPLES

1. A pile whose mass is ½ ton is driven 12 feet into the ground by 30 blows of a hammer whose mass is 2 tons falling 30 feet; prove that it would require 120 tons, in addition to the hammer, to be superposed on the pile to drive it slowly against the resistance of the ground (supposed to be uniform).

Prove that each movement of the pile takes 0·0228 seconds. (Greenhill's *Notes on Dynamics*, p. 27.)

2. If the resistance of the ground to the penetration of a pile is 5~~6~~4 tons' weight, prove that 10 blows of a hammer whose mass is 1½ ton falling 24 feet will drive a pile whose mass is ½ ton 5 feet into the ground.

Prove that each movement of the pile occupies 0·034 seconds.

3. A hammer whose mass is 2 pounds, moving with a velocity of 50 feet per second, strikes a nail whose mass is 1 ounce and drives it 1 inch into a fixed block of wood, the diameter of the nail being $\frac{1}{16}$ inch. From this prove that a rigid bullet whose mass is 1 ounce and diameter ½ inch moving with a velocity of 1500 ~~feet~~ per second will penetrate the block to a depth of 1·16 inches, supposing that the resistance of the wood to penetration is proportional to the cross-sectional area of the penetrating body.

Determine also the time occupied by the penetration in each case. (Greenhill, *ibid.* p. 28.)

Result. The nail takes $\frac{11}{3200}$ of a second, and the bullet $\frac{165}{128\times10^4}$.

(Observe that if we take $\frac{k}{100}$ to represent the resistance of the block to the nail, the resistance to the bullet will be $\frac{k}{4}$: resistances proportional to the squares of the diameters.)

4. If a hammer whose mass is 1 pound moving with a velocity of 34 feet per second strikes a nail whose mass is 1 ounce and drives it 1 inch into a fixed block of wood, prove that the mean value of the resistance of the wood to the nail is 204 pounds' weight, and that the time of penetration is $\frac{1}{192}$ sec. (Greenhill, *ibid.*)

41. **Hammer, Nail, and Block Problem.**—It will be a useful exercise in the study of the transformation of energy, as well as in the principles relating to impulse, momentum and work, to suppose that the block of wood, in the preceding examples, into which the nail is driven is moveable. Thus, with the data of Example 4, let us suppose that the block of wood has a mass of 68 pounds, and is free to move on a horizontal plane.

We may represent three succcessive stages in the phenomena, as in figures 75, 76, 77.

Fig. 75.

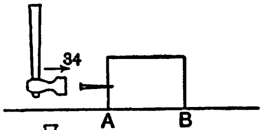


Fig. 76.

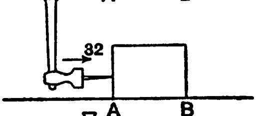
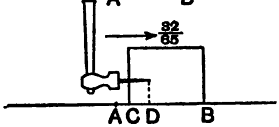


Fig. 77.



In fig. 75 the hammer head is represented as moving towards the nail with a velocity of $34\ {}^{f}/_{s}$, the nail and the block being at rest.

In fig. 76 the impulse between the hammer and nail is finished, and the hammer and nail are represented as travelling together with a velocity of $32\ {}^{f}/_{s}$ (which is deduced by the equation of impulse), while the block still remains at rest, the force produced on the block by the nail not being sufficiently great to produce any perceptible motion of the block during the extremely small time occupied by the blow between hammer and nail.

When this blow is finished, the pressure of the nail on the block will very gradually move the block, whose velocity will go on increasing until a stage is reached in which hammer, nail, and block will all travel with the same velocity. If v is this velocity, the momentum of the whole system will then be

$$(68 + 1 + \tfrac{1}{16})v$$

in pound-foot-second units, and this must be equal to the original momentum—viz. 1×34,

$$\therefore\ v = \tfrac{32}{65}\ {}^{f}/_{s}.$$

This is the state of affairs represented in fig. 77, the block

having in the meantime moved over a small distance AC. Beyond this stage the motion of the system does not concern us.

Let us exhibit in a table the successive velocities and kinetic energies of the system in these three stages.

Velocity.	Kinetic Energy of System.	Remarks.
34	$\frac{1 \times 34^2}{64} = 18\frac{1}{16}$	The diff. between these is lost to the system as heat.
32	$\frac{17}{16} \cdot \frac{32^2}{64} = 17$	The diff. between these does the work of crushing the block.
$\frac{32}{65}$	$(68 + \frac{17}{16}) \frac{32^2}{64 \times 65^2} = \frac{17}{65}$	

The original kinetic energy in the system (that of the hammer) was $18\frac{1}{16}$ foot-pounds' weight: when the impulse between the hammer and nail is ended, the kinetic energy of the system drops to 17 foot-pounds' weight, and the loss is converted into heat and sound. The final kinetic energy of the system drops to $\frac{17}{65}$ foot-pounds' weight; and the difference between this and 17 goes into the work of crushing the block.

Now the resistance to crushing by the nail was proved by the experiment in Example 4 (in which the block was *fixed*) to be 204 pounds' weight; so that if CD is the distance through which the nail penetrates the block, we have

$$204 \times CD = 17 - \tfrac{17}{65}$$

$$\therefore \quad CD = \tfrac{1}{12} \times \tfrac{64}{65} \text{ feet}$$

$$= \tfrac{64}{65} \text{ inch.}$$

What is the value of the distance AC through which the block has moved when the penetration is complete? This can be obtained by the equation of work and energy for the hammer and nail as one body from stage 2 to stage 3. The

hammer and nail have worked against the force 204 pounds' weight through the distance AD; their expenditure of kinetic energy in this interval is $\frac{17}{16}\left(32^2 - \frac{32^2}{65^2}\right)\frac{1}{64}$; so that, equating this to 204 AD, we have

$$AD = 1 - \frac{1}{65^2} \text{ inches};$$

and as CD is $\frac{64}{65}$ inches, we have $AC = \frac{1}{65} - \frac{1}{65^2}$ inches.

What is the time occupied by the nail in penetrating the wood? If t seconds is the time, equate 204 t to the gain of momentum of the block divided by g; hence

$$204\, t = \frac{68 \times \frac{32}{65}}{32} = \frac{68}{65},$$

$$\therefore \quad t = \tfrac{1}{195} \text{ sec.}$$

Take the following example in general symbols:

A hammer of weight W moving with a velocity v feet per second strikes a nail of weight w and drives it a distance of x feet into a wooden block which is held fixed; if the block is made moveable and has a weight P, prove that the nail will be driven through a distance $\frac{P}{W+P+w} \cdot x$, while the block will have moved through $\frac{P(W+w)}{(W+P+w)^2} \cdot x$, and the nail will have taken $\frac{2P(W+w)}{W(W+P+w)} \cdot \frac{x}{v}$ seconds to penetrate.

A good typical example of impulses and of the transformation of kinetic energy is furnished by two particles connected by a slack cord, one of the particles receiving a velocity. Let AB, (fig. 78), be a smooth horizontal table on which is placed a mass of weight P which is connected by a slack cord passing over a pulley at the edge, B, with a mass of weight Q lying on the ground. P is projected along the table with a given velocity, V; how high will Q be raised?

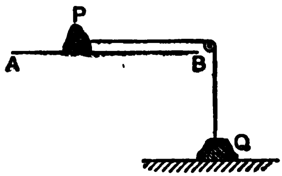


Fig. 78.

Here the initial kinetic energy of the system is $\frac{PV^2}{2g}$, and if Q is raised through a height x, the work done against gravity is $Q.x$; but it would be a fatal mistake to say that the kinetic energy $\frac{PV^2}{2g}$ is equal to this work. We shall see that the kinetic energy available for doing work is less than $\frac{PV^2}{2g}$.

When the cord becomes tight each mass receives an impulse, or blow, from the tension. Denote this impulse by B; and let v be the velocity with which Q is suddenly pulled off the ground; then v will also be the velocity of P along BA if the cord is (as we assume) inextensible; that is, P's original velocity, V, (which remains constant, as the table is smooth, until the instant of tightening) is suddenly converted into v; hence we have for P's motion

$$B = \frac{P(V-v)}{g}, \quad . \quad . \quad . \quad .$$

and for Q's

$$B = \frac{Qv}{g}, \quad . \quad . \quad . \quad . \quad .$$

so that

$$(P+Q)v = PV, \quad . \quad . \quad . \quad .$$

which shows that the sum of the momenta of the masses remains the same after the blow as it was before.

After the blow, then, the kinetic energy of the system is $\frac{(P+Q)v^2}{2g}$, i.e.

$$\frac{P^2V^2}{2g(P+Q)} \text{ is the kinetic energy available for work,} \quad . \quad (\alpha)$$

which is a fraction, $\frac{P}{P+Q}$, of the original kinetic energy, $\frac{PV^2}{2g}$.

Exactly as in Art. 36 in the discussion of Atwood's machine, in which it is pointed out that the work done on P by the tension of the cord after the blow is over is equal and

opposite to that done on Q by the tension, the sum of the two kinetic energies (α) after the blow is equal to the work done against gravity; therefore

$$\frac{P^2V^2}{2g(P+Q)}=Q.x \quad . \quad . \quad . \quad (\beta)$$

$$\therefore \; x=\frac{P^2V^2}{2gQ(P+Q)}.$$

When the blow is over, the tension in the cord assumes a finite value which is easily found, as at p. 83, from the equation $\frac{T}{P}=\frac{Q-T}{Q}$, $\therefore \; T=\frac{PQ}{P+Q}$.

Whenever, as in this example, a slack inextensible cord connects two particles, and this cord becomes suddenly tight, there is instantly a loss of *available* kinetic energy in the system; that is, a certain amount of the kinetic energy which existed before the tightening is instantly converted into the energy of motion of the *molecules* of the two bodies and radiated from these bodies in heat or sound into the surrounding air. Whenever a sudden tightening of a cord, or any kind of blow, occurs in a system (say, by collision) we must bridge over the time of the blow by applying

the equation of impulse and momentum,

and it is only after the new velocities which instantly succeed the blows have been calculated that we can employ

the equation of work and energy.

The only way in which a loss of available kinetic energy could be avoided in this example of motion is to give Q an upward velocity equal to that, V, with which P is moving, just at the instant at which the cord is about to become tight; but then our initial kinetic energy would have to be regarded as $(P+Q)\frac{V^2}{2g}$ and not $P\frac{V^2}{2g}$; and of course this kinetic energy would be equal to $Q.x$.

By the term *available* kinetic energy we mean kinetic energy which can be used for doing some desired work. Here, for

example, if the desired work is that of raising Q, all energy which will not help in this object is regarded as lost, wasted, or unavailable. If our object were the production of heat or sound, the energy previously regarded as wasted in the blow, viz. $\frac{PV^2}{2g}-\frac{P^2V^2}{2g(P+Q)}$, would be regarded as the useful part.

EXAMPLES

1. A mass of 8 kilogrammes on a smooth table is connected by a cord passing over a pulley at the edge with a mass of 12 kilogrammes lying on the ground, the cord being slack; the first is projected along the table with a velocity of 10 $^f/_s$; find
   (*a*) the distance through which the system will move before coming to rest for an instant,
   (*b*) the tension of the cord after the tightening of the cord.
   (*c*) the time between the tightening and the return motion.
   Result. 5 inches; $4\frac{4}{5}$ kilogrammes' weight; $\frac{5}{24}$ second. (Find (*c*) by *impulse and momentum.*)

2. A mass of 40 ounces on a rough horizontal plane ($\mu=\frac{1}{2}$) is connected with a mass of 10 ounces lying on the ground by a cord which has 5 feet of slack; the first mass is projected along the plane with a velocity of 20 $^f/_s$; find the three things required in the last example.
   The velocity of the mass on the table when it has moved over 5 feet —*i.e.* just before the cord tightens—is $4\sqrt{15}$ $^f/_s$; the principle of impulse and momentum shows that the two masses will start with a common velocity of $\frac{16}{5}\sqrt{15}$ $^f/_s$; so that the initial available kinetic energy is 120 foot-ounces' weight; the work to be done is the raising of the second mass and the overcoming of friction—*i.e.* $30x$ foot-ounces' weight, if the system moves through x feet; hence the required distance is

   4 feet.

   The tension after the tightening is 4 ounces' weight, and it remains at this value until the motion stops. There is no return motion; and the time between the tightening and the stoppage is $\frac{1}{6}\sqrt{15}$ seconds.

3. If in the last example the mass on the rough horizontal plane is 2 pounds, the hanging mass 3 pounds, the length of slack portion initially 8 feet, the coefficient of friction $\frac{1}{2}$, and the velocity of projection of the mass on the table 24 feet per second, discuss the motion.
   Result. After the tightening of the cord the system will move through 1 foot; the time occupied in this motion will be $\frac{1}{8}\sqrt{15}$ seconds, and the tension will be $\frac{3}{5}$ pounds' weight; in the return motion the tension will become $\frac{9}{5}$ pounds' weight.

4. If the magnitudes in the last example are, respectively, P, Q, a, μ, and V, find the height to which Q is raised.

Result. $$\frac{P^2}{(P+Q)(Q+\mu P)} \cdot \frac{V^2 - 2\mu g a}{2g}.$$

5. If in Atwood's machine (see fig. 59, p. 82) the mass P is 18 grammes, and it carries on its top a removeable bar whose mass is 8 grammes, while Q is a mass of 22 grammes, and the fixed ring through which P passes is 3 feet below A, the initial position of P, find the point to which P rises when it has come up again through the ring.

At starting the acceleration of P downwards is $\frac{26-22}{48} \cdot g$, i.e. $\frac{8}{3}$ f/ss, and its velocity in going down through the ring is 4 f/s. When the bar is picked off, the acceleration is reversed and becomes $\frac{22-18}{40} g$, i.e. $\frac{16}{5}$ f/ss. P will return to the ring with the velocity 4 f/s, and will experience a *blow* on taking up the bar from the ring. In this blow energy will be lost (*i.e.* transformed into the molecular form), and at the end of the blow the velocity of P upwards (and of Q downwards) will become $\frac{40}{48} \times 4$, or $\frac{10}{3}$ f/s, and P will ascend to a height of $\frac{25}{12}$ feet above the ring. The same succession of events will then follow, the successive heights attained diminishing in geometrical progression, the common ratio being $(\frac{40}{48})^2$, or $\frac{25}{36}$.

Another good example of impulse and energy is furnished by a *box of sand* suspended by parallel cords into which a shot is fired.

Let w be the weight of a bullet which is moving horizontally with a velocity V just as it is about to strike the box in a line passing through the centre of gravity of the box, the box hanging from two fixed points, P and Q (fig. 79), by two equal and parallel cords, PA and QB. In a small fraction of a second the bullet will come to rest inside the sand, having traversed a distance x within the box; and when the penetration of the sand is complete, the box and bullet will move with a common horizontal velocity, v. At this instant the two cords will have assumed the positions PC and QD, which very nearly coincide with their original positions, PA and QB, the small displacements AC and BD being equal and very nearly horizontal.

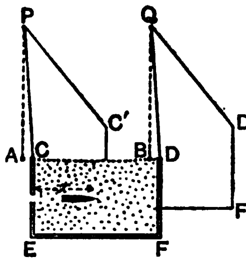


Fig. 79.

When the box and bullet go together with the common velocity v, the whole system rises to the position $C'E'F'D'$, exhausting thereby the kinetic energy which it had *immediately after the penetration of the sand was complete* in doing work against gravity.

Now if we measure the *vertical* height h, through which the box has been lifted, we can infer the value of V, the original velocity of the bullet.

For, during the time of penetration of the sand, the force, F, which the bullet exerts on the sand is equal and opposite to the force exerted by the sand on the bullet, so that the *impulse* received in this time is the same for both (see Art. 36). If we take F as constant, and if τ is the time of penetration of the sand, we have for the bullet

$$\frac{w(V-v)}{g}=F.\tau, \quad . \quad . \quad . \quad . \quad (1)$$

and for the box, whose weight is W,

$$\frac{Wv}{g}=F.\tau, \quad . \quad . \quad . \quad . \quad (2)$$

(F being taken in gravitation units); therefore $(W+w)v=wV$, as might have been foreseen;

$$\therefore v=\frac{w}{W+w}V. \quad . \quad . \quad . \quad . \quad (3)$$

The kinetic energy of the system, then, in the position $CEFD$ is $(W+w)\frac{v^2}{2g}$, or

$$\frac{w^2}{W+w}\cdot\frac{V^2}{2g}=\text{available kinetic energy}; \quad . \quad . \quad (4)$$

and (see Art. 35) the work done against gravity from the position $CEFD$ to the position $C'E'F'D'$ is $(W+w)h$; therefore

$$\frac{w^2}{W+w}\cdot\frac{V^2}{2g}=(W+w)h, \quad . \quad . \quad . \quad (5)$$

$$\therefore V=\frac{W+w}{w}\sqrt{2gh}, \quad . \quad . \quad . \quad (6)$$

which gives the required velocity.

By measuring the distance x through which the bullet has moved in the sand, we can find the value of the resistance, F, of the sand to the bullet.

For, when we consider what becomes of the kinetic energy which is transformed in the motion from the position $PABQ$ to the position $PCDQ$, we see that it has passed into heat by means of the crushing of the sand. The work of crushing the sand is $F \times x$; hence

$$\frac{wV^2}{2g} - \frac{w^2}{W+w} \cdot \frac{V^2}{2g} = F.x \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \; F = \frac{W(W+w)}{w} \cdot \frac{h}{x}, \text{ by (6)} . \quad (8)$$

The time, τ, occupied in the penetration of the sand can now be found from (2), and we get

$$\tau = \frac{2w}{W+w} \cdot \frac{x}{\sqrt{2gh}} . \quad . \quad . \quad . \quad . \quad (9)$$

The small distance AC moved over by the box while the bullet is penetrating the sand is

$$\frac{w}{W+w} . x, \quad . \quad . \quad . \quad . \quad (10)$$

as can be deduced by the equations of work and energy for bullet and box thus: the force F acts on the bullet through the distance $AC+x$, while it acts on the box only through the distance AC. Hence for the gain of kinetic energy by the box during the action of F we have

$$\frac{Wv^2}{2g} = F . AC,$$

and for the loss by the bullet,

$$\frac{w(V^2-v^2)}{2g} = F.(AC+x);$$

$$\therefore \; \frac{AC+x}{AC} = \frac{w(V^2-v^2)}{Wv^2} = \frac{V+v}{v}$$

$$\therefore \; AC = \frac{v}{V} . x = \frac{w}{W+w} . x.$$

Another interesting example of the same principles (impulse and momentum, work and energy) is furnished by a *gun from which a shot is fired.*

To reduce the figure to its simplest form, we shall represent (fig. 80) merely a tube containing a charge of powder from which a bullet is fired, the tube being, like the sand-box, suspended by two equal, parallel, and vertical cords, PA and QB, fig. 79.

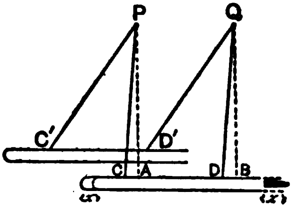


Fig. 80.

For the present we shall suppose the weight of the charge negligible when compared with W, the weight of the gun, and w, the weight of the shot.

When the charge is fired, there is generated a gas which at each instant during the passage of the bullet exerts the same pressure, F, on the shot as on the end of the barrel (in opposite senses, of course), so that when the shot is leaving the barrel the *impulse* of the powder pressure on the bullet is equal to that on the gun, since the *duration*, τ, of the pressure is the same for both shot and gun.

If, then, V is the velocity of the shot just as it is leaving the barrel, and v the backward velocity of the gun at this instant, and F is the mean value of the powder pressure during the time τ, we have

$$\frac{wV}{g} = F.\tau, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{Wv}{g} = F.\tau, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \; Wv = wV, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

i.e. the momentum of the gun is equal to the momentum of the shot. (Of course, this is true at each instant during the time τ.)

When the shot is leaving the muzzle, the cords will have moved very slightly and will be, suppose, PC and QD; *i.e.* the gun will have moved backwards through the distance AC, or BD. Denote this distance by x.

If l is the length of the barrel, the shot will have moved

forward through the distance $l-x$. (Of course *relatively to the barrel* it has moved through the distance l.)

Now observe particularly that the *impulse* of the powder pressure *is the same for shot and gun*, because the same force, F, acts on each for the *same time*; but the *work done* by the pressure on the shot is *not at all the same* as the work done by the pressure on the gun, because this force does not act through the *same distance* on both—the distances being $l-x$ and x. If the force acted through the same distance on both, the work done on each would be the same, *i.e.* the kinetic energy of the gun would be equal to the kinetic energy of the shot—or, in other words, the destructive power of the gun to its owners would be as great as that of the bullet to the enemy, because the mischief-working capacity of a missile is measured by its *kinetic energy* and not by its *momentum.*

If the rifle recoils and rises to the position $C'D'$ before coming to rest for an instant, and if the vertical height of $C'D'$ above CD is h, we have by work and energy

$$\frac{Wv^2}{2g}=Wh \quad \therefore\ v=\sqrt{2gh}, \quad . \quad . \quad (4)$$

and therefore $V=\frac{W}{w}\sqrt{2gh}=$ the muzzle velocity of the shot.

To find the mean value of the powder pressure, we have by work and energy

$$\frac{Wv^2}{2g}=F.x$$

$$\frac{wV^2}{2g}=F(l-x)$$

$$\therefore \quad \frac{x}{l-x}=\frac{v}{V}=\frac{w}{W} \quad \therefore\ x=\frac{w}{W+w}l, \quad . \quad (5)$$

$$F=(W+w)\frac{W}{w}\frac{h}{l} \quad . \quad . \quad . \quad (6)$$

and if τ is the duration of the explosion,

$$F.\tau=\frac{wV}{g},$$

$$\therefore\ \tau=\frac{wl}{W+w}\sqrt{\frac{2}{gh}} \quad . \quad . \quad . \quad (7)$$

If E and e are the kinetic energies of the bullet and gun when the bullet is leaving the muzzle, we have $E = \frac{wV^2}{2g}$ and $e = \frac{Wv^2}{2g}$, therefore $\frac{E}{e} = \frac{wV^2}{Wv^2} = \left(\frac{wV}{Wv}\right)^2 \cdot \frac{W}{w} = \frac{W}{w}$, which shows that the kinetic energies are *inversely* proportional to the weights of shot and gun. If S is the total amount of static energy in the powder charge—*i.e.* the amount that will be developed on explosion — we have $E + e = S$. Hence $E = \frac{W}{W+w} \cdot S$, so that the *square* of the velocity of the bullet is proportional to the weight of the powder.

EXAMPLES

1. A box of sand whose mass is 2000 pounds is suspended by two equal vertical cords, each 8 feet long, and a shot whose mass is 20 pounds is fired into it in a horizontal line passing through the centre of gravity of the box of sand, the shot remaining embedded; if the box recoils through a circular arc the length of whose chord is 6 feet, prove that the velocity of the shot is 1212 f/s.

If the shot penetrates 2 feet into the sand, prove that the mean resistance of the sand to the shot is 227250 pounds' weight, that the time of penetration is $\frac{1}{303}$ sec., and that during this time the box will have moved through $\frac{24}{101}$ inches. (Greenhill, *Notes on Dynamics*, p. 29.)

2. A rifle whose mass is 10 pounds is suspended horizontally by two equal parallel cords, and is observed to have risen 1 foot after firing a bullet whose mass is 1 ounce: the length of the barrel traversed by the bullet is 3 feet; find the velocity of the bullet, the time taken by the bullet to travel through the barrel, and the distance through which the rifle recoils in this time. (Greenhill, *ibid.*)

Result. 1280 f/s; $\frac{3}{644}$ sec.; $\frac{36}{161}$ inches.

3. If with a given powder charge a bullet fired vertically reaches a height h, what charge will be required to make the bullet reach double the height?

Ans. Four times the original charge.

4. Supposing that the penetration of a body of given material by a projectile of weight w moving with a velocity of v f/s is x feet when the body is fixed, prove that if a block made of this same material, of weight W, is free to move on a smooth horizontal plane and is a feet thick, the body will be perforated if

$$a < \frac{W}{W+w} \cdot x,$$

and that after passing through the body the projectile will retain a velocity

$$\frac{w+\sqrt{W^2-W(W+w)\dfrac{a}{x}}}{W+w} \cdot v.$$

(Greenhill, *ibid.*)

42. **Collision of Spheres.**—The case in which two spheres both moving in the line joining their centres come into collision and do not separate has been already discussed. There is only one magnitude to be found, namely, their common velocity after collision; and for this the equation of impulse and momentum (p. 122) suffices.

The spheres, however, in nearly all cases do separate after collision, so that there are two velocities, x and y (fig. 72, p. 119), to be found.

The necessary additional relation between x and y is obtained by an experimental law—viz.

the velocity of separation is a constant fraction of the velocity of approach.

This fraction (always denoted by e) depends on the substances of which the spheres are made. It never reaches the value *unity*—except, possibly, for the molecules of a gas which are in perpetual collision with each other inside the vessel containing the gas. The largest value of e has been observed in the collision of two glass balls, viz. $e=\frac{15}{16}$; next comes the value of e for two ivory balls, viz. $e=\frac{13}{16}$; for two of lead, $e=\frac{1}{5}$; two of cork, $\frac{2}{3}$, and almost the same value for two of cast iron.

Newton (*Principia*, Book I., Scholium) made a series of experiments for the determination of this coefficient by taking two spheres, A and B, of any materials, suspending them by two equally long threads from two fixed points, P and Q, in such a way that when they hang at rest the threads are both vertical and the bodies A and B just touch each other. By drawing out the threads from the vertical through equal angles at opposite sides and then letting them go, A and B will collide in their lowest (original) positions with equal and opposite velocities; each will be reflected and will rise to a certain measured vertical height above the lowest position. These heights will for each give the velocity with which it

was reflected (see p. 78); and in this way the above law is verified and e found for the bodies.

Like many other such experimental laws (*e.g.* the law of friction, p. 68) the constancy of e is only approximate, and the law must not be taken too seriously.

This coefficient e is called the *coefficient of restitution* of the bodies.

Now to apply the law. Take fig. 72, p. 119. The velocity of separation is $y-x$, and the velocity of approach is $v+v'$, so that

$$y-x=e(v+v'), \quad . \quad . \quad . \quad . \quad (1)$$

and this with the momentum equation

$$wx+w'y=wv-w'v' \quad . \quad . \quad . \quad (2)$$

determines both x and y.

If in fig. 72, p. 119, the ball w' was before collision moving in the same sense as w, and $v>v'$, the relative velocity of approach is $v-v'$, and our equations become

$$y-x=e(v-v'), \quad . \quad . \quad . \quad . \quad (3)$$

$$wx+w'y=wv+w'v'. \quad . \quad . \quad . \quad (4)$$

At every instant during the contact the resultant momentum remains the same. Hence if at the instant of maximum compression u is the common velocity of the spheres, we have

$$(w+w')u=wv+w'v', \quad . \quad . \quad . \quad (5)$$

assuming that v and v' were in the same sense.

Except in the ideal case in which $e=1$, there is always a loss of kinetic energy produced by the collision—*i.e.* a transformation of some of the original kinetic energy, $\frac{wv^2}{2g}+\frac{w'v'^2}{2g}$, into heat. This is evident to common-sense; for, if e is <1, the bodies separate in a more or less crushed condition, and some work must have been done to produce this crushing—exactly as work is done in the case of the shot and sand-box in crushing the sand (p. 133)—and this work never again restores kinetic energy to the system, but is the means of producing molecular motions, or heat which is ultimately given to the surrounding air.

The loss of kinetic energy is best calculated thus: Let

$E = \frac{wv^2}{2g} + \frac{w'v'^2}{2g}$ = kinetic energy of system before collision;

$E' = \frac{wx^2}{2g} + \frac{w'y^2}{2g}$ = kinetic energy after collision; then

$$E - E' = \frac{w(v^2 - x^2) + w'(v'^2 - y^2)}{2g} \quad . \quad . \quad (6)$$

$$= \frac{w(v - x)(v + x) + w'(v' - y)(v' + y)}{2g} \quad . \quad (7)$$

But from (4) we have $w'(v' - y) = -w(v - x)$; hence (7) becomes

$$2g(E - E') = w(v - x)(v + x - v' - y)$$

$$= w(v - x)(1 - e)(v - v'), \text{ by } (3) \quad . \quad (8)$$

Now from (3) and (4) we have

$$(w + w')x = wv + w'v' - ew'(v - v')$$

$$\therefore \; (w + w')(v - x) = w'(1 + e)(v - v') \quad . \quad . \quad (9)$$

Substituting this value of $v - x$ in (8), we have

$$E - E' = \frac{ww'}{2g(w + w')}(1 - e^2)(v - v')^2 \quad . \quad (10)$$

the right hand side being the energy transformed. This vanishes if $e = 1$.

If the particles were moving in opposite senses before collision, as in fig. 72, p. 119, the relative velocity of approach, $v - v'$, would become $v + v'$ in (10).

The blow sustained by each is given by the equation

$$B = \frac{w(v - x)}{g}$$

$$= \frac{ww'}{g(w + w')}(1 + e)(v - v') \quad . \quad (11)$$

If the two spheres were devoid of restitution, *i.e.* if $e = o$, the blow, B_o, would be given by

$$B_o = \frac{ww'}{g(w + w')}(v - v');$$

hence we see that when there is separation, the blow is greater than B_0, and

$$B=(1+e)B_0 \quad . \quad . \quad . \quad . \quad (12)$$

From (10) we see that the energy transformed is equal to $\frac{1}{2}B(1-e)(v-v')$; or if a and s are the relative velocities of approach and separation, ***both measured in the same sense,***

$$E-E'=B\,.\,\frac{a-s}{2}. \quad . \quad . \quad . \quad (13)$$

If a ball strikes a *plane*, whether fixed or moving, the experimental law (p. 139) connecting the velocity of recoil with that of approach must, of course, be taken as holding.

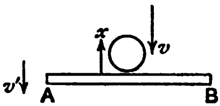


Fig. 81.

Thus, if AB (fig. 81) represents a plane which is being mechanically moved with a velocity v' perpendicularly to AB, while a ball overtakes the plane with a normal velocity v, and if x (which may be in the sense represented or in the reverse) is the velocity of reflection of the ball, we have, with the arrows as represented in the figure,

$$\text{velocity of separation}=x+v',$$
$$\text{,, approach}=v-v',$$
$$\therefore \quad x+v'=e(v-v'). \quad . \quad . \quad (14)$$

Of course if the plane AB is moved by some mechanism, there is no question about the alteration of v' by the impact, and hence no momentum equation is required.

If w is the weight of the ball, and the blow given by the plane $=B$,

$$B=\frac{w}{g}(1+e)(v-v'). \quad . \quad . \quad . \quad (15)$$

EXAMPLES

1. Two spheres whose masses are 3 ounces and 2 ounces are moving in the line of centres towards each other with velocities 50 f/s and 60 f/s, respectively; their coefficient of restitution is $\frac{1}{2}$; find their velocities after collision, and the amount of kinetic energy transformed.

Result. They are both driven back, the first with a velocity of 16 f/s, and the second with a velocity of 39 f/s. The kinetic energy transformed is $170\frac{5}{32}$ foot-ounces' weight.

2. In the last case find the magnitude of the blow sustained by each, and if the duration of contact is $\frac{1}{84}$ sec., find the mean value of the pressure between them.

Result. $B = 6\frac{3}{16}$ foot-second-ounces' weight units; mean value of pressure $= 396$ ounces' weight.

3. Two spheres whose masses are 10 and 15 grammes are moving in the line of centres with velocities of 80 c/s and 60 c/s in the same sense, the first overtaking the second; find the velocities after collision and the quantity of kinetic energy transformed, their coefficient of restitution being $\frac{1}{4}$.

Result. The velocities are, respectively 65 c/s and 70 c/s both in the sense of the original motion; the energy transformed amounts to 1125 ergs. (*N.B.* g is taken as 981 c/ss.)

4. If in the last the duration of contact is $\frac{1}{884}$ sec., find the value of the mean force between them.

Result. 100 grammes' weight.

5. Three spheres, A, B, C, of equal masses are placed in this order at any distances in a right line on a smooth horizontal table; $e = \frac{1}{2}$ for A and B, and $e = \frac{1}{3}$ for B and C; A is projected towards B with any velocity; how many collisions will take place between them?

Ans. Only two.

6. If in the last the masses of A and B are equal while the mass of C is twice that of either, how many collisions will take place, and what will be the final velocities of A, B, C?

Ans. Three collisions, and if v is the original velocity of A, the velocities will be $\frac{v}{8}$, $\frac{5v}{24}$, $\frac{v}{3}$.

7. If in Q. 5 the masses of A, B, C are proportional to 3, 2, 6, how many collisions will there be, and what will be the final velocities?

Ans. Four collisions; $\frac{4}{25}v$, $\frac{3}{10}v$, $\frac{8}{25}v$.

8. Two spheres, of masses A and C, are placed on a horizontal table; between them is placed a sphere of mass B; A is projected towards B, and a velocity thus communicated to C; find the mass of B so that C's velocity may be greatest.

Result. B must be the geometric mean between A and C.

9. Two marbles are placed inside a smooth groove in the shape of a closed curve of any form, the distance between them, measured along the groove being a, and the whole length of the groove being l; one of the particles is projected along the groove with a velocity v so that the distance gone over before the first collision is a; find the time interval up to the first collision and the intervals between the successive collisions.

Result. $\frac{a}{v}$, $\frac{l}{ev}$, $\frac{l}{e^2v}$, $\frac{l}{e^3v}$, etc.

10. Two spheres, of masses A and B, are projected towards each other on a smooth horizontal table with given velocities; can they after collision exchange velocities exactly?

Ans. Not unless $e=1$.

11. A circular hoop of mass M and radius r rests on a smooth horizontal table; if a marble of mass m is projected from the centre with a velocity v, prove that the interval between the first and second impacts is $\frac{2}{e} \cdot \frac{r}{v}$, and find the ultimate velocity of the marble and hoop.

Result. $\frac{m}{M+m} \cdot v$.

12. A plane surface is moved with a constant velocity of 20 f/s normally to itself, and is struck normally by a ball moving in the same sense with 60 f/s; $e=\frac{1}{2}$ for ball and plane; find the motion of the ball after impact.

Result. The ball is reduced to rest by the impact.

13. If in the last the plane is moving with 20 f/s to meet the ball, what is the motion of the ball after impact?

Ans. Its original velocity is exactly reversed.

The case of the *oblique collision* of two smooth spheres is very easily disposed of. A numerical case will at once make the thing clear.

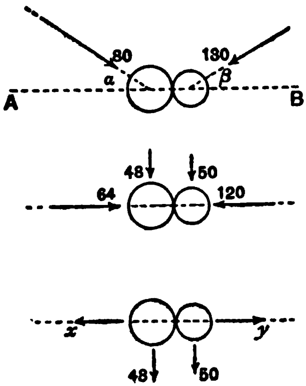


Fig. 82.

Let AB, fig. 82, be the line joining the centres of two smooth spheres at the instant of collision, one moving with 80 f/s in a line making an angle α such that $\tan\alpha=\frac{3}{4}$ with AB, and the other moving with 130 f/s in a line making an angle β, such that $\tan\beta=\frac{5}{12}$ with AB; let the masses of these spheres be, respectively, 7 and 5 grammes, and let $e=\frac{1}{2}$ for them. Required the magnitudes and directions of their velocities after collision.

Resolve each of the velocities 80 and 130 into two components along and perpendicular to the line of centres. Thus we have the second figure.

Now since the surfaces are smooth, *neither* of the components perpendicular to AB will be altered by the collision: it is only the component velocities along AB that will be altered; so that the first will after collision be still moving with a velocity of 48 f/s perpendicularly to AB, and the second with a velocity 50 f/s in this direction. Let the unknown component velocities in the line of centres be x, y; and thus we have the third figure.

Hence the problem is only one in direct collision: two spheres whose masses are 7 and 5 grammes are moving towards each other in the line of centres with velocities of 64 and 120 f/s respectively; e is $\frac{1}{2}$; find their velocities after collision.

Our equations for x and y are

$$\begin{aligned} 7x - 5y &= -448 + 600 = 152, \\ x + y &= \tfrac{1}{2}(64 + 120) = 92, \\ \therefore\ x &= 51;\ y = 41. \end{aligned}$$

Hence the line of resultant motion of the first is the diagonal of a rectangle whose adjacent sides along and perpendicular to BA are 51 and 48; and the line of resultant motion of the other the diagonal of a rectangle whose sides are 41 and 50.

The *resultant* velocities are $\sqrt{51^2 + 48^2}$ and $\sqrt{41^2 + 50^2}$.

The blow received by each sphere is wholly in the line AB, and its magnitude is

$$\frac{7(64 + 51)}{g} = \frac{805}{32} \text{ foot-second-grammes' weight units.}$$

The loss of kinetic energy to the system is due wholly to the motion in the line of centres; for the kinetic energy of the first body before collision is $\frac{7 \times 80^2}{64}$, or $\frac{7}{64}(64^2 + 48^2)$; and after collision it is $\frac{7}{64}(x^2 + 48^2)$. Hence the loss is

$$\frac{7(64^2 - 51^2) + 5(120^2 - 41^2)}{64} \text{ foot-grammes' weight.}$$

It is not worth while to express the results of oblique collision in a general formula. The procedure in every case is exactly

that indicated above—viz. *resolve the velocity of each sphere into two components, one along and the other transverse to the line of centres; assume each transverse component unaltered, and calculate the other components by the principles of direct collision.*

The case in which a ball strikes *obliquely* a plane either fixed or moving is also easily disposed of by the same principles. Thus (fig. 83) suppose that a ball moving in the line CP with a velocity v strikes a fixed plane AB at the point P and is reflected. Let Pn be the normal to the plane at P, and let x and y be the components of v (called the *velocity of incidence*) along and perpendicular to the plane. Then, assuming no friction between the ball and the plane,

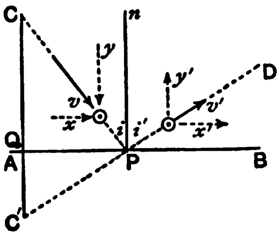


Fig. 83.

the component x will be the same after incidence as before, while if y' is the new normal component, we have

$$y' = ey,$$

so that the components of the velocity after striking the plane are

x along the plane, and
ey perpendicular to the plane,

and the new line of motion, PD, is the diagonal of a rectangle whose sides are x along PB and ey along Pn.

The angle CPn is called the *angle of incidence*, and DPn the *angle of reflection*. Denoting these by i and i' and the velocity of reflection (in PD) by v', we have

$$v' \cos i' = e \,.\, v \cos i, \qquad (1)$$

$$v' \sin i' = v \sin i, \qquad (2)$$

$$\therefore \tan i' = \frac{1}{e} \,.\, \tan i, \qquad (3)$$

$$v' = v \sqrt{\sin^2 i + e^2 \cos^2 i}. \qquad (4)$$

Hence the angle i' is always greater than i, except in the ideal case in which $e = 1$, and in this case the angles of incidence and reflection are equal—as in the case of an incident and reflected ray of light.

Equation (3) supplies the solution of the following problem: a ball is to be projected from a given point, C, to strike a fixed plane, AB, and to pass after reflection through a given point, D; find the requisite direction, CP, of projection.

From C draw CQ perpendicular to AB, and produce it to C' so that $QC' = e \cdot QC$: join D to C'; then DC cuts AB in the point P, and CP is the required direction. The proof is obvious.

And in exactly the same way we solve the problem: a ball is to be projected from a given point, P, fig. 84, and after reflection from any number of fixed planes, AB, BC, CD, is to pass through a given point, Q; find the requisite direction, Pn, of projection.

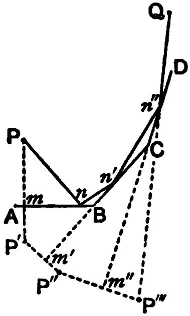


Fig. 84.

Let e, e', e'', be the coefficients of rebound of the ball and the planes AB, BC, CD; draw Pm perpendicular to AB and produce it to P' so that $mP' = e \cdot mP$; from P' draw Pm' perpendicular to BC, and produce it to P'' so that $m'P'' = e' \cdot m'P'$; from P'' draw $P''m''$ perpendicular to CD and take $m''P''' = e'' \cdot m''P''$; let QP''' cut CD in n''; let $n''P''$ cut BC in n'; let $n'P'$ cut AB in n; then the path of the required path is $Pnn'n''Q$.

This supposes the radius of the ball to be negligible; if it be not so, we must replace the given planes AB, BC, CD by planes parallel to them at a distance r, where r is the radius of the ball: we thus obtain the path of the centre of the ball.

EXAMPLES

1. $ABCD$ is a square; a sphere whose mass is 15 grammes is moving from A to B with a velocity of 35 c/s, and one whose mass is 14 grammes is moving from C to B with a velocity of 40 c/s, the line of centres at the instant of collision being BC; if $e = \frac{1}{1}\frac{3}{8}$ find the velocities of the spheres after collision.

   Result. The sphere that was moving in AB moves in the diagonal DB produced through B with a velocity of $35\sqrt{2}$ c/s, while the other sphere continues its motion CB with a velocity of $2\frac{1}{2}$ c/s.

2. If the duration of contact in this case is $\frac{1}{317}$ sec., what is the mean value of the force exerted between the spheres?

   Ans. 175 grammes' weight.

3. AB (fig. 82) is the line of centres of two spheres at the moment of collision; the masses are 10 and 8 grammes; the first is moving with a velocity of 153 c/s in a line making with the lower side of AB an angle whose tangent is $\frac{8}{15}$, and the second with a velocity of 90 c/s in a line making with the upper side of AB an angle whose tangent is $\frac{4}{3}$; e is $\frac{2}{3}$ for the spheres; find their speeds and lines of motion after collision, and the magnitude of their pressure, supposing the duration of contact to be $\frac{1}{1962}$ secs. Find also the amount of transformed kinetic energy.

Result. After collision the first sphere has a velocity of 72 c/s perpendicular to AB, and one of 5 c/s along BA; the components of velocity for the second are 72 c/s perpendicular to AB and 121 c/s along AB. The mean pressure is 2800 grammes' weight, and the energy transformed is 44,100 ergs.

4. In the last what will be the distance between the centres of the spheres after t seconds?

Ans. $18\sqrt{113}\,t$ centimètres.

5. All the data being as in Q. 3 except that the magnitude of the velocity of the sphere whose mass is 8 grammes is not given, what must this velocity be in order that after collision the other sphere may move at right angles to AB?

Ans. $78\frac{3}{4}$ c/s.

6. All the data being as in Q. 3 except that the masses of the spheres are not given, what must be the ratio of their masses so that after reflection they may both move in the same right line?

Ans. 13 : 22.

7. A marble is placed at a given point on the bottom of a rectangular box whose sides are vertical; show how the marble must be projected against one side so that after reflection from each of the others it may return to its original position.

(Apply the construction in p. 47, the point Q being the same as P.)

8. If in the last e is the same for the marble and each side, show that the path of the marble is a parallelogram.

9. A ball moving with a velocity v strikes a moving plane at an angle of incidence i, the plane having a velocity u in the direction of its normal; find the angle of reflection.

Result. If i' is the angle of reflection, $\tan i' = \dfrac{v \sin i}{ev \cos i \pm (1+e)u}$.

EXAMINATION ON CHAPTER VIII

1. Describe the state of affairs in the collision of two spheres from the moment of first contact to the moment of separation.

2. What are the two important facts relating to the pressure (or various pressures) between the spheres during the collision?

3. Draw roughly a curve representing the values of the pressure at various instants during the collision.

What, in the figure, represents the whole impulse on either sphere during contact? (Area.)

4. Enunciate the principle of the *conservation of momentum* for the spheres.

5. What about the sum of the kinetic energies of the spheres *before* collision and the same sum *after* separation?

What becomes of the difference?

6. What are the two principles to be used in calculating the distance through which a pile is driven into soft ground by a hammer blow?

7. If the foot of the pile meets a rock how are these principles affected?

8. What is meant by *available energy*? Is the whole of the kinetic energy of the hammer available for the work of forcing the pile into the ground?

9. If a hammer drives a nail into a moveable block is there anything conserved after the penetration is complete? (Momentum.)

10. In this case what kinetic energy is converted into heat before the nail begins to penetrate, and what kinetic energy is used up in crushing the block?

11. If a body lying on the ground is connected by a slack cord with a body lying on a horizontal table and the latter body is projected along the table, is its kinetic energy available for raising the mass on the ground?

12. Which is the more readily applicable principle in the case of forces *very suddenly* applied—the principle of *impulse and momentum* or the principle of *work and energy*?

13. Trace the application of these principles in the case of a shot fired into a suspended box of sand.

14. In the case in which a gun fires a shot why is the *momentum* of the shot equal to that of the gun? Why are their *kinetic energies* vastly different? If a rifle of mass 10 lbs. fires a 2-ounce bullet, how many times greater than the kinetic energy of the rifle is that of the bullet?

15. What is meant by the *coefficient of restitution* of the spheres? Give the numerical values of some coefficients of restitution.

16. How much kinetic energy is transformed and radiated during the collision of two spheres?

17. State shortly the way in which the velocities after collision are calculated in the case of two obliquely-colliding spheres. How much kinetic energy is transformed during their collision? (Only that corresponding to motion in line of centres.)

18. If a ball projected from a given point is to strike a fixed plane, how do you find the direction of projection so that, on reflection, the ball shall pass through a given point?

19. What is meant by *angle of incidence* and the *angle of reflection*? If $e=1$ for the ball and the plane, what is the relation between these angles? If e is not equal to 1, which is the greater?

CHAPTER IX

NON-CONCURRENT FORCES

43. *Concurrent forces* mean forces whose lines of action all pass through a common point. In the previous chapters we have confined our attention to such forces. We now proceed to consider a body of any size whatever—no longer a *particle*—acted upon by forces whose lines of action do not pass through the same point, but we shall assume these lines of action to lie all in the *same plane*.

Such forces have evidently a single resultant; and it can obviously be found by taking the forces two at a time, producing their lines of action to meet, and at the point of meeting constructing the resultant of the two forces by the parallelogram law. Thus, suppose that we have any number of *coplanar* forces (*i.e.* forces whose lines of action all lie in the same plane), P_1, P_2, P_3, P_4, P_5, . . .; produce the lines of action of P_1 and P_2 to meet; take their resultant, R_{12}; in the same way find the resultant of R_{12} and P_3; let this be R_{123}; find the resultant, R_{1234}, of R_{123} and P_4; and so on, until all the forces have been taken; then we shall have finally the resultant, R, of the whole set.

This merely shows *how* the thing could be done; but the actual process of thus combining forces in pairs would be very long and tedious, and there are better ways of arriving at the final result, as we shall explain in this chapter.

Let us begin by taking a special case which naturally presents itself to us—viz. the case of two *parallel forces*, *i.e.* forces whose lines of action are parallel.

Two Parallel Forces.—Let a body be acted upon by two forces whose magnitudes are P and Q, acting in the lines marked AP and BQ in fig. 85. If we attempt to find their

resultant by producing their lines of action to meet, we encounter the difficulty that the point of meeting is at infinity. Hence we must adopt another method; and that method consists in ***replacing the two parallel forces by two non-parallel forces which are completely equivalent to them.*** Thus, let the line AP represent the magnitude of P and BQ the magnitude of Q; draw AB; at A and B apply two equal and opposite forces in the line AB, these forces having any magnitude whatever, AF towards the left and BF towards the right. These two equal forces, F, F, acting in the same line in opposite senses produce no effect on the body if the body is ***rigid***, as we assume.

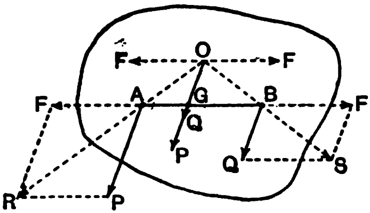


Fig. 85.

Now take the resultant, AR, of AP and AF, and also the resultant, BS, of BQ and BF, and we see that the two given parallel forces AP and BQ are equivalent to the two non-parallel forces AR and BS. Let these latter meet in O, and if O is rigidly connected with the body, we can let AR and BS each act at O. If O is outside the physical limits of the body, we must imagine it to be rigidly connected with the body by a thin rigid membrane, or something of the kind. Re-resolve these two forces at O into their original components F and P for the first, parallel and equal to AF and AP; and F and Q for the second, parallel and equal to BF and BQ. Then we have the two equal and directly opposed forces F, F at O, which may be struck out as cancelling each other; and, in addition, we have at O the two forçes P and Q acting together in the same line, OG, parallel to their original lines of action, AP and BQ.

Hence the resultant of the two parallel forces is

$$P + Q,$$

the forces being, observe, in the same sense. Parallel forces acting in the same sense are called ***like*** parallel forces; those in opposite senses are called ***unlike*** parallel forces.

The line of action, OG, of the resultant is defined thus: since the triangle AGO is similar to RPA,

$$\frac{AG}{GO}=\frac{RP}{PA}=\frac{F}{P};$$

similarly

$$\frac{BG}{GO}=\frac{SQ}{QB}=\frac{F}{Q};$$

$$\therefore \quad \frac{AG}{GB}=\frac{Q}{P}, \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

or

$$P\times AG=Q\times BG \quad . \quad . \quad . \quad (\beta)$$

Hence—the resultant of two like parallel forces, P and Q, is equal to their sum, $P+Q$; its line of action is between the lines of action of the two forces, and it divides any line drawn across their lines of action into two segments inversely proportional to the forces.

Cor. 1.—The resultant of two equal and like parallel forces is equal to twice either force, and its line of action is midway between them.

Cor. 2.—Any force, P, can be broken up into two parallel forces, T and $P-T$ the distances of whose lines of action from that of P are inversely as the magnitudes of the components, T and $P-T$.

For, let AP (fig. 86) represent P in magnitude and line of action; let BT represent any parallel force, T, in the same sense; and let C be a point on BA such that

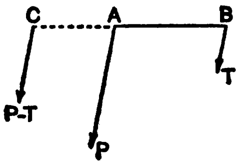


Fig. 86.

$$(P-T)\times AC=T\times AB;$$

then by the result (β) above we see that the resultant of the force T acting at B and a parallel force $P-T$ at C is the force P acting at A; that is, P can be replaced by T at B and $P-T$ at C.

Now let us find the resultant of two *unlike* parallel forces. We may proceed exactly as in the case of like forces, or thus, making use of the result of Cor. 2.

Let the unlike forces be P acting at A, fig. 87, and Q acting in the opposite sense at B. Suppose P to be the greater.

Break up P into a force BQ' equal and opposite to Q acting at B, and a force equal to $P-Q$ acting at G. Then by Cor. 2., G is such that $(P-Q) \times AG = Q \times BA$, or

$$P \times GA = Q \times GB.$$

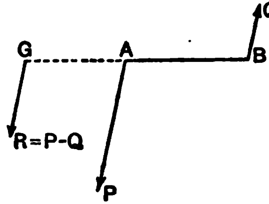


Fig. 87.

Since the two opposed forces Q, Q acting at B cancel each other the resultant of P and Q is

$$P-Q$$

acting at G.

Hence—the resultant of two unlike parallel forces, P and Q, is equal to their difference, $P-Q$; its line of action is outside the lines of action of the two forces, at the side of the greater; and it divides any line drawn across their lines of action into segments inversely proportional to the forces.

Observe that in fig. 87 AB is *any* line drawn across the lines of action of P and Q, and the segments of AB made by the point G are GA and GB.

If the magnitudes of the two unlike parallel forces are equal, *i.e.* if $P=Q$, their resultant is *a force of zero magnitude acting at infinity*; for if $P=Q$, we have $GA=GB$, which requires G to be at infinity.

We must now occupy some time in studying this curious result. Of course if we *actually* replace the two unlike forces P, Q by their resultant, $P-Q$, we must *rigidly attach* its point, G, of application (or its line of action) to the body which is acted upon by P and Q. Before dealing directly with two equal parallel forces acting in opposite senses along any two lines, we shall prove the *property of moments* for parallel forces, whether like or unlike; that is, we shall show that—

> *the (algebraic) sum of the moments of two parallel forces about any point in their plane is the same in magnitude and sense as the moment of their resultant about the point.*

Let P, Q (fig. 88) be two like parallel forces acting on a body, and O any point in their plane. From O draw OA and

OB perpendicular to P and Q; then the point, G, in which their resultant, R, intersects AB is obtained by the relation

$$P \times GB = Q \times GB.$$

Now the moment of P about O is $P \times OA$, and its sense is counterclockwise; the moment of Q about O is $Q \times OB$, and its sense is clockwise; hence the sum of the moments about O in the counterclockwise sense is $P \times OA - Q \times OB$, which is

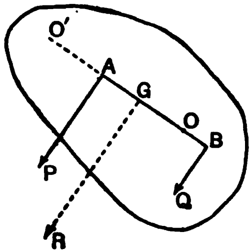


Fig. 88.

$$P \times (OG + GA) - Q \times (GB - OG),$$

or $(P + Q) \times OG + P \times GA - Q \times GB$,

i.e. $(P + Q) \times OG$,

since $P \times GA - Q \times GB = 0$; and this is the moment of R about O.

In the same way, if O is taken anywhere else, as at O', the sum of the moments of P and Q about it is the same in magnitude and sense as the moment of R about it.

When the forces P and Q are in opposite senses (fig. 89) the same result holds. The resultant is now $P - Q$, if P is the greater, and the point, G, in which it cuts AB is still found from the relation

$$P \times GA = Q \times GB.$$

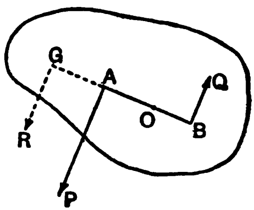


Fig. 89.

The sum of the moments about O is now $P \times OA + Q \times OB$, or $P \times (OG - GA) + Q \times (GB - OG)$, or $(P - Q) \times OG$, as before.

As a numerical example of two unlike parallel forces, let $P = 1001$, $Q = 1000$, pounds' weight, suppose; then $R = 1$ pound weight, and we have

$$GA = 1000AB;$$

thus G is very distant from A and B, and if O is any point in the body, the perpendicular, OG, from O on the line of action of R is very great, so that in the moment of R about O we have *a very small force*, R, *multiplied by a very great perpendicular, or lever arm*, OG; and this moment is, of

course, equal to $P \times OA + Q \times OB$, a very considerable number of foot-pounds' weight. In the same way if $P = 1000{\cdot}001$, and $Q = 1000$, R will be $\frac{1}{1000}$ pound weight, and AG will be $1000000AB$, so that we have an extremely small force acting in a line extremely distant from the body, while the moment of this very small and distant force about O is $1000 \times AB + \frac{1}{1000} . OA$.

It is clear, then, that, *however small R may be, it has a finite, and perhaps very large, moment about every point in the body*—viz. the sum of the moments of the two unlike parallel forces themselves about the point.

Pass now to the case in which $P = Q$, and we have $R = 0$, $AG = \infty$. Looked at in this way, we have actually *a zero force acting at infinity*; but looked at from the point of view of the property of moments, we see that the moment of R about any point in the body is a perfectly finite amount, viz. the sum of the moments of the two equal and unlike forces themselves.

Now when the two unlike forces P and Q are equal, *they have the same moment in magnitude and sense about all points in their plane.*

For, if we denote the sum of these moments about O in fig. 89 by $\overleftarrow{M_o)}$, we have

$$\begin{aligned} \overleftarrow{M_o)} &= P \times OA + P \times OB \\ &= P \times AB. \end{aligned}$$

This moment is the same whatever point, O, is chosen on AB. If O is taken outside the lines of action of the forces, say at the side of the point G in fig. 89, and we draw the line OAB perpendicular to the lines of action as before, we have

$$\begin{aligned} \overleftarrow{M_o)} &= -P \times OA + P \times OB \\ &= P \times AB, \end{aligned}$$

wherein there is nothing depending on the position of the point O.

Hence, then, the very important result stated above—viz. *two equal and opposite parallel forces acting in different lines have a moment which is constant both in magnitude and in sense about all points in their plane.*

The converse of this is also an important result—viz. *if two*

forces have a moment which is constant in magnitude and sense about all points in their plane, the forces must be equal and opposite parallel forces. For, unless they are equal and opposite parallel forces, they have a resultant of finite magnitude acting in a line which is not at infinity, and such a force must have variable moments about points which are more or less distant from its line of action.

Two equal and opposite parallel forces acting in different lines are called a *couple*. If h is the perpendicular distance between their lines of action, h is called the *arm* of the couple; and if P is the magnitude of each of the forces, the product $P.h$ is the moment of the couple.

44. **Resultant of any number of Parallel Forces.**—Let any given parallel forces, P_1, P_2, P_3, . . . (fig. 90) act on a rigid body, and let it be required to find the magnitude and line of action of their resultant.

Since the magnitude of the resultant of two parallel forces P and Q is equal to their *algebraic* sum, it is obvious that by taking some two and combining their resultant with a third force, and so on, *the magnitude of the resultant of any number of parallel forces is the algebraic sum of the forces.*

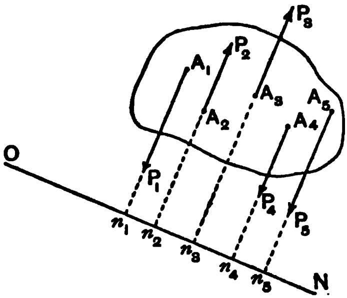


Fig. 90.

Thus, with the senses of the forces as shown in fig. 90, the magnitude of the resultant, R, is

$$P_1 - P_2 - P_3 + P_4 + P_5. \quad . \quad . \quad . \quad (1)$$

To find the line in which it acts, employ the principle of moments—*i.e.* that its moment about any point is equal to the algebraic sum of the moments of the given forces about that point.

Thus, let O be any point in the plane of the forces, and let the perpendiculars from O on the lines of action of P_1, P_2, P_3 . . . be On_1, On_2, On_3 . . . ; then if p is

the perpendicular from O on the line of action of R, we have

$$R.p = P_1 \times On_1 - P_2 \times On_2 - P_3 \times On_3 + P_4 \times On_4 + P_5 \times On_5, \quad (2)$$

which gives p, and therefore the line of action of R.

Let, for example, $P_1 = 10$ pounds' weight, $P_2 = 10$, $P_3 = 12$, $P_4 = 4$, $P_5 = 20$, and $On_1 = 6$ inches, $On_2 = 8$, $On_3 = 10$, $On_4 = 14$, $On_5 = 18$; then $R = 12$, and $p = 23$ inches, so that if we take $ON = 23$ inches, and at N draw a line in the direction of the forces, this is the line of action of R.

In algebraic notation, if the *algebraic* sum of the forces is denoted by ΣP we have

$$R = \Sigma P, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and if p denotes the perpendicular from O on the line of action of any one, P, of the parallel forces, the distance, ON, of the line of action of R from O is given by the equation

$$ON = \frac{\Sigma P.p}{\Sigma P}. \quad . \quad . \quad . \quad . \quad (4)$$

Equation (2) or (4) gives us the result that—*the distance of the line of action of the resultant of any number of parallel forces from any point is obtained by dividing the sum of the moments (with their proper signs) of the given forces about the point by the sum of the forces (with their proper signs).*

EXAMPLES

1. Find the magnitude and line of action of the resultant of two like parallel forces whose magnitudes are 50 and 30 pounds' weight, the distance between their lines of action being 2 feet.

Result. The resultant is 80 pounds' weight, its line of action is between those of the forces, and is 9 inches from the line of action of the greater.

2. In the last find the sum of the moments of the forces about a point between the lines of action and distant 16 inches from the greater force.

Result. 560 inch-pounds' weight.

3. In the same case find the position of a point about which the algebraic sum of the moments of the two forces is 1680 inch-pounds' weight.

Result. Any point on a line parallel to the force 50, and 30 inches distant from it; or any point on a line parallel to the 50 at a distance of 12 inches from it. (These moments are, of course, of different signs.)

4. Two forces have a constant clockwise moment of 100 inch-pounds' weight about all points in their plane; represent them in a figure.

Two equal and opposite parallel forces of 100 pounds' weight with a distance of 1 inch between their lines of action; or two of 50 with a distance of 2 inches; or two of 1 pound weight with a distance of 100 inches; or, generally, two of magnitude x pounds' weight with a distance of $\frac{100}{x}$ inches—producing a clockwise moment.

45. **Equilibrium of Two Couples.**—*If a rigid body is acted upon by two couples of equal and opposite moments, the body is in equilibrium.*

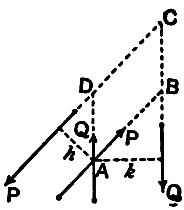


Fig. 91.

Let there be two equal and opposite parallel forces, P, P, fig. 91, with an arm h, and two equal and opposite parallel forces, Q, Q, with an arm k, such that

$$P.h = Q.k,$$

the sense of the couple P, P being counter-clockwise, while that of Q, Q is clockwise; let their lines of action when produced form the parallelogram $ABCD$.

Then since the angles at B and D are equal, we have

$$\frac{k}{AB} = \frac{h}{AD}; \text{ or } \frac{k}{h} = \frac{AB}{AD};$$

but, by hypothesis,

$$\frac{k}{h} = \frac{P}{Q};$$

$$\therefore \quad \frac{P}{Q} = \frac{AB}{AD},$$

i.e. the forces P and Q may be represented by the sides of the parallelogram $ABCD$.

Now the resultant of P and Q, each acting at A, is represented by the line AC, and the resultant of the remaining P and Q, each applied at C, is represented by the line CA; and these resultants are equal and opposite in the same line; hence the four forces constituting the two couples are in equilibrium. Q.E.D.

An obvious consequence of this result is that *a couple can be changed into any other couple which has the same moment*

and sense as the given one. For example, two equal and opposite parallel forces, each equal to 20 pounds' weight, with an arm of 5 inches, and a clockwise moment, acting on a rigid body, can be changed into two forces each of 50 pounds' weight with an arm of 2 inches provided that the sense of this latter couple is also clockwise; and, generally, the couple of forces (P, P), fig. 92, with arm a can be replaced by the couple (Q, Q) with arm b, provided that

$$Q \cdot b = P \cdot a.$$

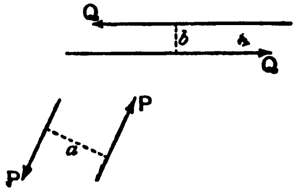


Fig. 92.

For, by the above, if we reverse each of the forces Q, Q, we obtain a couple whose moment is equal to that of P, P, while its sense is opposite, and these couples are in equilibrium; therefore, etc. In other words, *the only things essential to a couple are its sense and the magnitude of its moment.*

46. *Composition of Couples.*—Couples acting in one plane (coplanar couples) are compounded by simple addition and subtraction of their moments. To see this, let us take a numerical example. Let the sense of a couple (clockwise or counterclockwise) be denoted, as before, by a bent arrow; and suppose a rigid body to be acted upon by the following couples: $100\circlearrowright$ inch-pounds' weight, $80\circlearrowleft$, $96\circlearrowright$, and $60\circlearrowleft$. Now each of these can be converted into a couple of forces having an equivalent moment. Let each of them be converted into a couple having any common arm—say 4 inches; then $100\circlearrowright$ is equivalent to two opposite parallel forces of 25 pounds' weight at the ends of an arm, AB, of 4 inches. Denote this result thus,

$$
\begin{array}{lrl}
 & 100\circlearrowright & = \uparrow 25,\ 25\downarrow;\ \text{arm } 4 \\
\text{also} & 80\circlearrowleft & = \downarrow 20,\ 20\uparrow,\ \ ,,\ \ ,, \\
 & 96\circlearrowright & = \uparrow 24,\ 24\downarrow,\ \ ,,\ \ ,, \\
 & 60\circlearrowleft & = \downarrow 15,\ 15\uparrow,\ \ ,,\ \ ,, \\
\therefore & \text{sum of moments} & = \uparrow 14,\ 14\downarrow,\ \ ,,\ \ ,,
\end{array}
$$

because at the point A of the arm AB we have the resultant force $25-20+24-15$, and an opposite force at B; that is, we have the single couple $\uparrow 14$, $14\downarrow$ with the arm 4, whose moment is $\overset{\frown}{56\downarrow}$ inch-pounds' weight. This is, of course, the algebraic sum, $100-80+96-60$, of the moments of the given forces.

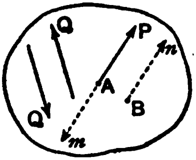


Fig. 93.

If a couple is combined with a force acting in a given line, the resultant is simply the same force transferred to a parallel line. For, let P be force acting in the line AP, fig. 93, and let a couple (Q, Q) whose moment is $\overset{\leftarrow}{M})$ be combined with P; change this couple into a couple (P, P) whose arm is therefore $\frac{M}{P}$. Move this couple into the position in which one of its forces, Am, is directly opposed to the given force P, the other force being represented by Bn. Now since Am is equal to P, the two forces P and Am cancel, and we are left with the force Bn, which is equal and parallel to P. Hence the couple (Q, Q) and the force P result in the force Bn, or P. Therefore, etc.

Cor.—A force and a couple acting on a rigid body cannot produce equilibrium.

47. **To find the Resultant of any System of Forces.**—When we are given any system of coplanar forces acting on a rigid body, the resultant can be found by the long process described in p. 150; but it can be far more simply found by calculation as follows:

The resultant has—

$\left.\begin{array}{l}(a) \text{ the same component along every line}\\ (b) \text{ the same moment about every point}\end{array}\right\}$ as the given forces themselves.

These two properties enable us to lay down the resultant in magnitude, line of action, and sense.

A numerical example will make this clear.

A rigid body is acted upon by forces along the sides and diagonals of a rectangle $ABCD$ (fig. 94) as follows: 29 pounds' weight from A to D, 30 from C to D, 16 from C to B, 6 from

B to A, 39 from A to C, and 26 from D to B; the side AB is 12 inches long, and AD is 5 inches; find the magnitude and line of action of the resultant of these forces.

Call X the total component of the forces in the direction of AB, and Y the total component in the direction of AD. Then

$$X = -6 - 30 + 39 \times \tfrac{12}{13} + 26 \times \tfrac{12}{13} = 24,$$
$$Y = 29 - 16 + 39 \times \tfrac{5}{13} - 26 \times \tfrac{5}{13} = 18,$$

which give the resultant, R, equal to $\sqrt{X^2 + Y^2}$, or 30 pounds' weight. The values of X and Y give the magnitude, direction, and sense of R, but not its line of action. Thus, draw any lines, EF, EH, parallel to AB and AD; on the first measure off a length EF representing 24 pounds' weight, and on the second measure off a length EH representing 18 pounds' weight. (We may, of course, measure off these lengths on AB and AD themselves.) Complete the rectangle $EFGH$. Then R is parallel to EG and in the sense EG; but of course EG is not its line of action.

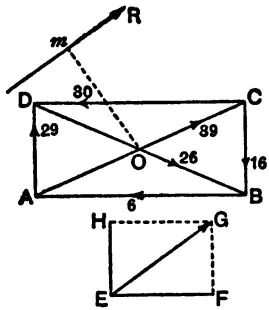


Fig. 94.

Now employ the principle (b) above, and choose O, the intersection of the diagonals of the rectangle $ABCD$, as the point about which moments are to be taken. Let $\widehat{M_o}\downarrow$ denote the sum of the moments of the forces about O; then

$$\widehat{M_o}\downarrow = 29 \times 6 - 30 \times \tfrac{5}{2} + 16 \times 6 + 6 \times \tfrac{5}{2} = 210 \text{ inch-pounds' weight}$$

Now this is the moment of R about O; and as $R = 30$ pounds' weight, if p is the perpendicular from O on its line of action, we have

$$30 \times p = 210 \quad \therefore p = 7 \text{ inches.}$$

Hence from O draw a line perpendicular to the direction EG, and measure off a length Om equal to 7 inches; at m draw a line parallel to EG; this line is the line of action of R. Observe that we measure the length Om equal to 7 inches towards the *upper* side of O, not the lower, because we know

that the moment of R about O is *clockwise*. Since R acts in the sense EG, it could not act at the lower side of O; for, if it did, its moment about O would be counterclockwise.

As further exercises take the following:

1. In fig. 94 let the force 29 in AD be replaced by a force 54 acting in the sense DA, and let the 16 in CB be replaced by a force 4 acting in the sense BC, no other change being made. Find the resultant.

Result. The resultant is a force of 51 pounds' weight, and its line of action is thus found: along EF, which is parallel to AB, measure a length EF to represent 24, produce HE through E to H' so that EH' represents 45; complete the rectangle $EH'G'F$; then R acts parallel to EG' and in the sense EG'; from O drop a perpendicular on EG', and on this perpendicular measure off a length, On, equal to 8 inches; then R acts in the line through n parallel to EG'.

2. In the same figure let the forces in AC and BD remain unaltered, while—

force from D to $A=25$; force from B to $C=20$; force from B to $A=84$; force from D to $C=24$.

Here the resultant vanishes altogether, since the forces have no component along any line, and no moment about O. These forces are in equilibrium.

3. ACB is a right-angled triangle whose right angle is C; AC is 4 inches, CB is 3 inches; a force of 112 pounds' weight acts from A to C, a force of 45 from C to B, and one of 85 from A to B; find their resultant completely.

Result. The resultant is a force of 204 pounds' weight in a line making $\tan^{-1}\frac{8}{15}$ with AC, at a perpendicular distance of 1 inch from C, its components along AC and CB being 180 and 96, and its moment about C is the same as that of the force 85 in AB.

4. In fig. 94 let the side $AB=15$ inches and $AD=8$ inches. Find the magnitude and line of action of the resultant of the following forces: 260 pounds' weight acting from A to B, 92 from C to B, 195 from C to D, 52 from A to D, 34 from A to C, and 51 from D to B.

Result. The resultant is a force of 148 pounds' weight acting in a line making $\tan^{-1}\frac{12}{35}$ with AB, at a perpendicular of 5 inches from O, its components in the senses AB and DA being 140 and 48 pounds' weight, respectively.

5. If two different systems of forces have the same moments about any three points A, B, C, the two systems must have the same resultant.

6. Hence if a system of forces has moments L, M, N about any three points A, B, C, and if Δ is the area of the triangle ABC, show that the resultant is the same as that of three forces $\frac{a.L}{2\Delta}$, $\frac{b.M}{2\Delta}$, $\frac{c.N}{2\Delta}$ acting along BC, CA, AB, these sides being a, b, c, respectively.

EXAMINATION ON CHAPTER IX

1. What is meant by coplanar forces? Why can they be replaced, in general, by a single force?

2. What is the rule for compounding two *like* parallel forces? What is the rule for *unlike* parallel forces?

3. Prove that the sum of the moments of any two parallel forces about any point in their plane is equal to the moment of their resultant about that point.

4. Define a couple. Explain the meaning of expressing the resultant of two equal unlike parallel forces as a *zero force acting at infinity*.

5. What is the fundamental property of a couple? (Constant moment about all points.) What is meant by the moment of a couple?

6. If any given system of coplanar forces has the same moment about all points in their plane, what do we know about their resultant?

7. If two unlike parallel forces whose magnitudes are 40 and 30 pounds' weight have moments of 160 and 180 inch-pounds' weight, both clockwise, about a certain point, what is the distance of the line of action of their resultant from that point? (34 inches.)

8. Prove that the only things essential to a couple are the magnitude and sense of its moment, and that it may be replaced by any other one having the same things.

9. What is the resultant of any number of given couples?

10. Explain how the magnitude and line of action of the resultant of *any* given system of coplanar forces are found.

CHAPTER X

EQUILIBRIUM OF RIGID BODIES

48. **Conditions of Equilibrium of a Rigid Body.**—When a rigid body is acted upon by any coplanar forces, the necessary and sufficient condition that it shall be in equilibrium is that the Resultant of the forces absolutely vanishes. In other words,

(*a*) the forces must have no component along any line, and also

(*b*) the forces must have a zero moment about every point.

The condition (*a*) would not of itself be sufficient, because it would be satisfied if the forces reduced to a couple; hence (*b*) must be added.

Two very special and simple cases of equilibrium, owing to their very frequent occurrence, must be pointed out.

Two Forces.—When a body is in equilibrium under the action of only *two* forces, what do we know about them?

They must be equal and opposite in the same right line.

Three Forces.—When a body is in equilibrium under the action of only *three* forces, what do we know about them?

Their lines of action must meet in a point. This is necessary because each one must be equal and opposite to the resultant of the other two, and for this purpose they must meet in a point. The result is seen also by the principle (*b*) above; for if we take moments about the point of meeting of two of them, we have the moment of the third about this point equal to zero—*i.e.* the third passes through the point. Meeting in a point includes parallelism: if two of the forces are parallel, the third must be parallel to them.

49. **Action at a smooth Axis.**—Before entering on the consideration of the equilibrium of rigid bodies in special cases, it is well to consider the nature of the action exerted on a body which, while acted upon by any forces, is moveable about a smooth cylindrical axis fixed in the body.

Let fig. 95 represent a body of any form in which there is fixed a smooth axis, A. There may be contact between the body and the surface of this axis all the way round, or contact merely along a line parallel to the axis. In the case of the joint, pin, or axis which connects the legs of an ordinary compass there is usually contact between the pin and a leg of the compass all round the pin; but the joint may be so worn that the leg and the pin really touch each other along a mere line. Suppose that there is contact all over the axis, and let fig. 96 represent a cross section of the axis. Imagine the whole surface of the axis broken up into very small elements, mn, nr, rs, . . . Then mn may be considered as a little plane surface, and the force exerted by the axis on the body over this surface is (since there is no friction) normal to mn at its middle point. This force passes, therefore, through the centre of the axis, as represented by the arrow. Similarly the action produced by the axis on the body over the little plane surface nr acts normally to this element at its middle point: this force, again, passes through the centre of the axis, and it may not be equal to the force on mn. In this way we see that each element of the surface of the body round the axis is acted upon by a force which passes through the centre of the axis.

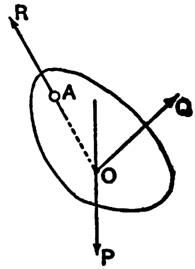


Fig. 95.

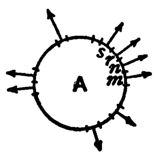


Fig. 96.

Now, *however variable in intensity these forces may be, they must have a single resultant passing through the centre of the axis.* This resultant *may assume any direction whatever*, but it must pass through the centre of the axis.

Moreover, this must also be the case when the axis is worn and contact takes place at a single line, (or point) p, as in fig. 97.

In all cases, therefore—

> *the resultant action exerted on a body by a smooth axis consists of a single force passing through the centre of the axis, and it may have any direction whatever through this centre.*

The direction which it will take depends on the other forces acting on the body. Thus, suppose that the body in fig. 95 is kept at rest by two given forces, P and Q, in addition to the pressure on the axis. Then, since the body is kept at rest by *three* forces, the lines of action of these must meet in a point; so that if P and Q meet at O, the reaction, R, of the axis on the body must pass through O. It must also pass through the centre of the axis; therefore its line of action is completely determined; and R must, of course, be equal and opposite to the resultant of P and Q.

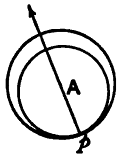


Fig. 97.

If the axis, A, is *rough*, in addition to the normal forces acting in fig. 96 on the various small elements of surface, there will be forces of friction which act all round the axis, and the resultant of these may act anywhere—perhaps in a line very distant from the centre of the axis; so that—

> *the action of a rough axis on a body moveable round it may be a force whose line of action is very distant from the centre of the axis.*

As a simple illustration of the foregoing principles, take the equilibrium of a *cantilever*, fig. 98. AB is a horizontal bar whose end A is fixed by a smooth pin vertically below A; CD is a bar jointed at C to the bar AB, its end D being fixed by a smooth pin; a load W is suspended from B, and the weights of the bars may be neglected in comparison with W; find the forces exerted on the pins at A, C, and D.

Consider the separate equilibrium of the bar CD. This bar is kept in equilibrium by only *two* forces—those exerted on it by the pins C and D (its weight being neglected). These two forces must be equal and opposite, and therefore they must act along the bar—*i.e.* in the line CD.

What forces keep the bar AB at rest? *Three* forces—viz.

W, the reaction of the pin A, and the reaction of the pin C, which last acts in the line DC. These three must meet in a point, which point must be that in which DC produced meets the line of action of W. Denote this point by O. Then the action at A must act in the line OA. The only possible senses of these forces are those of the arrows drawn from the point O. Hence the bar CD acts on AB in the line and sense CO, and of course (since action and reaction are equal and opposite) the action, R, of AB on CD is in the sense CD. The pin D is acting on the bar CD with the force R in the sense DC. Let S be the action of the pin A on the bar AB. Now, considering the equilibrium of the bar AB—*i.e.* of the three forces W, R, S acting through O, we may use either the law of sines or the triangle of forces. The triangle ADO has its sides parallel to W, R, and S; hence

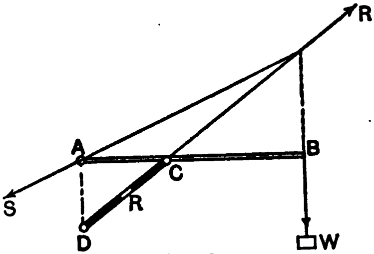


Fig. 98.

$$W : R : S = AD : DO : OA; \text{ that is}$$

$$R = \frac{OD}{DA} \cdot W \text{ and } S = \frac{OA}{AD} \cdot W,$$

so that the values of the pressures R and S are known in terms of W.

Supposing, for example, that $AB = 24$ inches, $AC = 12$, $AD = 5$; then since by similar triangles $\frac{AC}{CB} = \frac{DC}{CO}$, we have $CO = 13$, $\therefore DO = 26$. Also

$$AO^2 = AD^2 + DO^2 - 2AD \cdot DO \cos ADC, \therefore AO = \sqrt{601}, \text{ and}$$

$$R = \frac{26}{5} W, \; S = \frac{\sqrt{601}}{5} W.$$

We proceed now to illustrate the two general conditions (a) and (b) at the head of this article by taking some simple examples of the equilibrium of bodies which may be regarded as rigid.

A bar AB (fig. 99) of negligible weight is freely moveable round a smooth fixed horizontal axis at F, and has two masses whose weights are P and Q suspended from its ends; find the relation between P and Q for equilibrium, and the pressure on the axis at F. Let $AF=a$, $BF=b$. The bar AB is in equilibrium under the action of three forces, P, Q, and the reaction, R, at F. Since two of these forces, P and Q, are parallel, the third must be parallel to them; hence R is vertical. Resolving forces vertically, we have

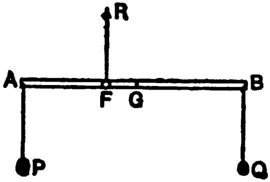


Fig. 99.

$$R-P-Q=0, \therefore R=P+Q.$$

Again, take moments about F, and we have $P.a=Q.b$, which is the relation between the two weights and the arms a and b.

Thus if $a=6$ inches and $b=8$ inches, and if $Q=18$ ounces, P must be 24 ounces, and $R=42$ ounces' weight. The bar AB is called a *lever*, and the fixed point F is called the *fulcrum*.

If the lever itself has weight, W, which is not negligible, and if this weight acts at a point G (called the *centre of gravity* of the lever) whose distance from the fulcrum is c, the equation of moments about F is

$$P.a=Q.b+W.c,$$

and $R=P+Q+W$.

Thus, if $a=6$ inches, $b=8$, and $c=2$, while $W=4$ ounces and $Q=14$ ounces, then P must be 20 ounces, and $R=38$ ounces' weight.

The *common balance* is a lever having, of course, weight, while its centre of gravity, G, coincides with the fulcrum, and the arms a and b are of equal length. In this case $P=Q$.

If the centre of gravity of the common balance coincides with the fulcrum, while the arms a and b are unequal, the weights P and Q are unequal, and the machine is a *false balance*.

In this case let a given body be weighed by suspending it

from A; then if its weight is W, it will be balanced by suspending a body of weight W_1 from B, such that

$$W_1 = \frac{a}{b} W. \qquad . \quad . \quad . \quad . \quad (1)$$

Now remove the body W from A and suspend it from B, balancing it again by suspending a body from A. If the weight of this latter is W_2, we have

$$W_2 = \frac{b}{a} W. \qquad . \quad . \quad . \quad . \quad (2)$$

The weights W_1 and W_2 are the *apparent* weights of the body whose true weight is W. Combining the results (1) and (2) by multiplication, we have

$$W = \sqrt{W_1 . W_2},$$

so that the true weight of the body is the geometric mean of its apparent weights in a false balance.

A common crowbar furnishes also an example of this kind of lever. AB (fig. 100) is a bar which rests at a point F on the top of some fixed body, such as a log of wood or a stone block, while the end B is inserted between the ground and the under surface of a heavy body, W, which we desire to lift or tilt up from the ground; a pressure P, called the *effort*, is applied at A to the bar, and this pressure has to balance the *resistance*, S, of the body W at B.

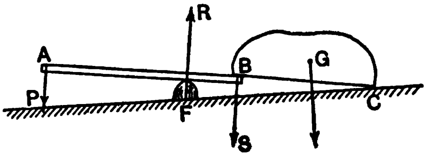


Fig. 100.

In this case F is the fulcrum of the lever, and if (as we may usually do) we neglect the weight of the crowbar itself, we have the bar in equilibrium—or in a state of *slow motion* which may be treated as equilibrium—under the action of the three forces P, S, and the resistance, R, at the fulcrum.

The relation between the effort and the resistance is obtained by taking moments about F for the equilibrium of the bar: thus,

moment of P about F = moment of S about F.

The forces P, R, S must, of course, either meet in a point or be parallel. Usually we may take them as parallel—though the crowbar may be so placed and used that S is not parallel to P: we might, for example, apply the effort obliquely instead of perpendicularly to the bar. If the forces are parallel, the arm of P about F is AF, and the arm of S is BF, so that

$$P \times AF = S \times BF,$$

$$\therefore \quad S = P.\frac{AF}{BF}.$$

If $AF = 4.BF$, for example, an effort applied at A will just overcome a resistance of 4 times the magnitude at B.

The ratio, $\frac{S}{P}$, of the resistance to the effort in a machine is called the *mechanical advantage* of the machine.

The crowbar may be so used that the fulcrum is outside the lines of action of the effort and the resistance, as in fig. 101. By moments again about the fulcrum, F, we have

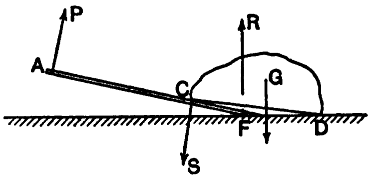


Fig. 101.

$$S = P.\frac{AF}{CF},$$

and in this case the pressure, R, at F is equal to $S - P$.

The Wheel and Axle, or *Windlass*.—On an axis AB (fig. 102) the ends of which are supported in some fixed structure is rigidly mounted a cylinder C of large radius, a, which forms one solid piece with another cylinder, D, of much smaller radius, r; a rope is coiled round D, and at the free end of this rope is applied a resistance R (usually a bucket filled with water which we desire to raise from a well, or a load of coal); another rope is coiled round the cylinder C, and to the free end of this

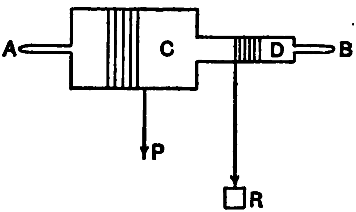


Fig. 102.

is applied the *effort* (see p. 115) P which is to overcome the resistance R.

If the friction at A and B produced by the support of the machine is negligible, we may consider the machine as kept in equilibrium by R, P, and the supporting forces at A and B, which latter forces have no moment about the line AB. Taking moments about AB, we have

$$P.a = R.r$$

$$\therefore\ P = R\frac{r}{a}.$$

Thus, if a is 10 times r, $P=\frac{1}{10}R$, so that to overcome any resistance an effort of only $\frac{1}{10}$ of its amount is required.

The *mechanical advantage* of any machine is the ratio of the resistance overcome to the necessary effort; and in this case the mechanical advantage is 10. Generally, the mechanical advantage of the wheel and axle is $\frac{a}{r}$. Sometimes the cylinder C is reduced to a simple wheel; and sometimes the machine has the form of fig. 103, in which the effort P is applied by hand to a handle, H, rigidly attached to the axis of rotation, AB. If the radius, BH, of the circle described by the handle is a, while r is the radius of the cylinder C, we have, as before, by moments about AB,

$$P.a = R.r.$$

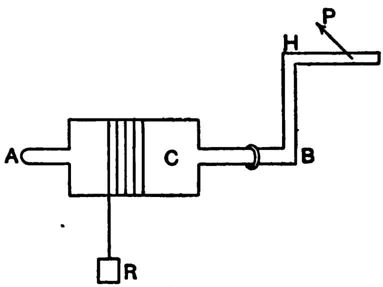


Fig. 103.

The machine is sometimes used in the form (fig. 104) of a *differential windlass*, in which a rope passing round a moveable pulley, M, is coiled in opposite senses round the cylinders C and D, while the resistance, R, is applied to the pulley. When the handle is turned, the rope winds *on* C and *off* D. Let the radius of C be r, that of D being r', while $BH=a$. Then the tension in the rope is the

same throughout (if friction can be neglected) and equal to $\frac{1}{2}$ R; and if we consider the equilibrium of the machine

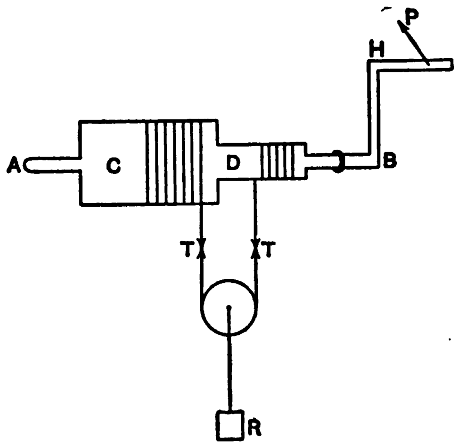


Fig. 104.

$ACDB$, the sum of the moments of the two tensions applied to it about AB, in the sense opposite to that of P, is

$$T(r-r'), \text{ or } \tfrac{1}{2}R(r-r').$$

Equating this to the moment of P, we have

$$P.a=\tfrac{1}{2}R(r-r').$$

The mechanical advantage of this differential windlass is $\dfrac{2a}{r-r'}$; so that, by making r' nearly equal to r, we can obtain a mechanical advantage as great as we please. If, for example, r is $1\frac{1}{2}$ feet, and $r'=1\frac{1}{4}$ feet, while a is 2 feet, we have

$$R=16P.$$

The disadvantage of making r' nearly equal to r is that the pulley M is then raised very slowly.

The Wedge.—The wedge (fig. 105) is an isosceles triangular

prism which is employed for splitting wood. We shall assume that the wedge is in contact with the wood at two points, A, B, symmetrically situated at the sides of the wedge. If there is no friction, the resistances at A and B will be two forces normal to the surface of contact. Let P be a pressure applied at the base of the wedge, and 2α the vertical angle of the wedge. Then, for equilibrium, we have, by resolving forces in the direction of P,

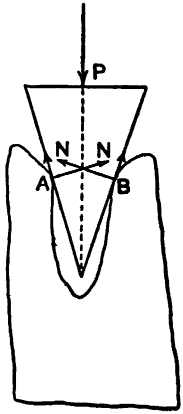


Fig. 105.

$$P = 2N \sin \alpha.$$

If there is friction, and the wedge is just about to enter the wood, the forces of friction at A and B being each μN, we have

$$P = 2N(\sin \alpha + \mu \cos \alpha).$$

Arrangements of Pulleys.—Combinations of pulleys are sometimes employed for the purpose of raising heavy loads. Thus, suppose that A and B (fig. 106) are two blocks to each of which are attached pulleys, the same rope passing *over* the upper system and *under* the corresponding pulleys of the lower system, the free end of the rope being pulled by a force P, while the body, W, to be raised is attached to the end of the lower block. The other end of the rope, after passing round the uppermost pulley of the lower block is attached to the end of the upper block.

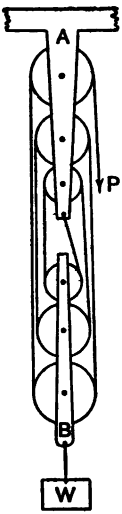


Fig. 106.

Then, the tension of the rope throughout being assumed to be constant and therefore (see p. 49) equal to P, if there are n pulleys in the lower block, each of these will be acted upon upwards by the force $2P$; so that, if we consider the equilibrium of the lower block and the body to be raised, we have the total upward force on this system equal to $2nP$. This must, for equilibrium or uniform motion, be equal to the total weight; and if the weights of the pulleys and block are negligible in comparison with W, we have

$$2nP = W.$$

If W' is the total weight of the block B and its pulleys, and if this is not negligible, we have

$$2nP = W + W'.$$

We have, of course, assumed that all the portions of the cord (*plies*, as they are called) are parallel and vertical.

If, as in fig. 107, we have an arrangement in which each pulley is suspended from the beam A, and if we can neglect the weight of each pulley, the tension of the rope round the pulleys B and C will be P; the tension of the rope round D will therefore be $2P$; the tension of the rope round E will be 2^2P; and if there were another pulley below E, the tension of its rope would be 2^3P; and so on. Hence if there were n *moveable* pulleys, the tension of the rope which passes round the last would be $2^{n-1}P$; and if we now consider the equilibrium of this last pulley and W, we have twice this tension equal to W; *i.e.*

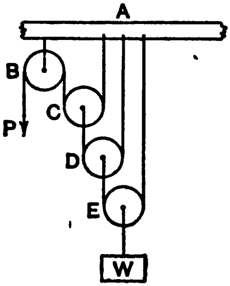
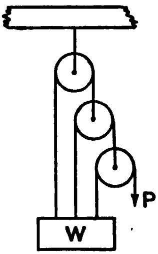


Fig. 107.

$$2^n P = W.$$

Finally, if, as in fig. 108, the rope passing round each pulley is attached to the body W, the tension of the rope round the lowest is P; round the next above $2P$; round the next 2^2P; and round the highest $2^{n-1}P$, where n is the whole number of pulleys. Now the total upward force acting on W is $P + 2P + 2^2P + \ldots + 2^{n-1}P$, that is, $(2^n - 1)P$, so that

$$(2^n - 1)P = W.$$



Fig, 108.

If the weights of the pulleys have to be taken into account, there is no difficulty in calculating the tensions in the various plies; but the result is not worth recording in general symbols.

Trains of Toothed Wheels.—Let a (fig. 109) be a toothed wheel of radius r_1 moveable round a fixed horizontal axis fitted with a handle to which an effort, P, is

applied; let this wheel gear with a larger wheel, A_1, of radius R_1 fixed to a horizontal axis on which is also rigidly mounted a wheel, a_2, of radius r_2; let a_2 gear with another wheel, A_2, of radius R_2, moveable round a fixed horizontal axis on which is rigidly mounted a cylinder, C, having coiled round it a rope from whose free extremity is suspended a body of weight W.

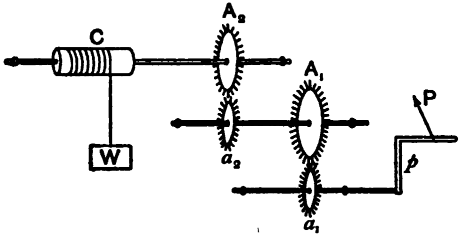


Fig. 109.

It is required to find the relation between the effort P and the resistance W, when the system is just about to move, or in uniform motion.

Suppose the pressure between a_1 and A_1 to be S_1; then if p is the lever arm of P about the axis of a_1, we have the equilibrium of this wheel

$$P.p = S_1 . r_1 \quad . \quad . \quad . \quad . \quad (1)$$

Again, if S_2 is the pressure between a_2 and A_2, we have for the equilibrium of the rigid system a_2 and A_1, by moments about its axis,

$$S_1 . R_1 = S_2 . r_2 \quad . \quad . \quad . \quad (2)$$

Finally for the equilibrium of the rigid system C and A_2, if c is the radius of C,

$$S_2 . R_2 = W . c \quad . \quad . \quad . \quad (3)$$

Multiplying the left sides and the right sides of (1), (2), (3) together,

$$P.p . R_1 . R_2 = W . c . r_1 . r_2$$

$$\therefore \quad P = W \frac{c}{p} \cdot \frac{r_1 r_2}{R_1 R_2}.$$

The wheels a_1, a_2 are called *drivers*, and A_1, A_2 are called *followers*.

If there are n drivers and n followers, the relation becomes

$$P = W \frac{c}{p} \cdot \frac{r_1 r_2 \ . \ . \ . \ r_n}{R_1 R_2 \ . \ . \ . \ R_n}.$$

If the drivers have all the same radius, r, and the followers all the same radius, R,

$$P = W\frac{c}{p} \cdot \left(\frac{r}{R}\right)^n.$$

[Friction on axles is, of course, neglected.]

Rack and Pinion.—Let there be a straight rod, A, fig. 110, provided with teeth which gear with a system of teeth projecting from the circumference of a wheel moveable round a fixed axis at its centre, this wheel being rigidly attached to a handle at which an effort, P, is applied perpendicularly, the resistance, R, being applied at one extremity of the toothed rod. This rod is called a ***rack***, and the toothed wheel a ***pinion***. The rack is capable of sliding freely along some fixed smooth surface parallel to its own direction, but of no other motion.

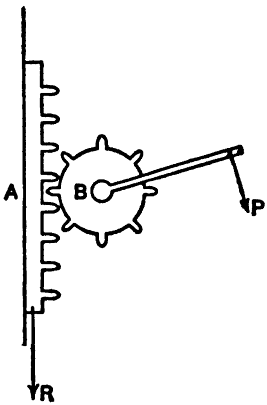


Fig. 110.

Let r be the radius of the pinion and p the length of the handle. Then if R includes the weight of the rack, the pressure between the rack and the pinion is R, so that for the equilibrium of the pinion we have simply

$$P.p = R.r.$$

EXAMPLES

1. A body when suspended from one arm of a balance appears to have a mass of $10\frac{5}{8}$ ounces, and when suspended from the other arm a mass of $10\frac{1}{8}$; find its true mass and the ratio of the arms.

Result. True mass = 10.374 ounces; ratio = 1.012 : 1.

2. There is a uniform lever, AB, 50 centimètres long, whose fulcrum does not coincide with its centre of gravity, and whose mass is 200 grammes; a mass of 106 grammes suspended from A is just balanced by a mass of 94 grammes suspended from B; when the first mass is suspended from B, it requires a mass of 119 suspended from A to balance the lever;

find the lengths of the arms and the distance of the centre of gravity from the fulcrum.

Result. The arms are 26 and 24 cms. long, and the centre of gravity is $\frac{1}{2}$ cm. from the fulcrum.

3. If the oar of a boat is 8 feet long and the rowlock is $2\frac{1}{2}$ feet from the handle, compare with the pull of the oarsman the resistance of the water and of the boat to the oar.

Result. These forces are proportional to 11, 5, and 16.

4. A straight bar, AB, 40 centimètres long is found to balance about a knife edge 18 centimètres from A; a mass of 110 grammes is suspended from B, and a mass P (to be weighed) is suspended from A. It is now found that the fulcrum must be placed at a distance of 10 centimètres from A. The mass P is next suspended from B, while the 110 mass is suspended from A, and it is found that the fulcrum must be moved to a distance of 29 centimètres from A. Find P and the mass of the bar.

Result. $P=490$ grammes, mass of bar$=200$.

5. AB is a uniform bar 20 inches long whose mass is 10 ounces; from it are suspended masses of 3, 4, and 15 ounces at distances 4, 8, 16 inches, respectively, from A; find the position of a knife edge about which the bar will balance horizontally.

Result. At a distance of 12 inches from A.

6. If a tradesman uses a false balance whose weight may be neglected, and puts the substance to be weighed as often into the one pan as into the other, show that he must lose by the transaction, and find his loss per cent.

Let a and b be the lengths of the arms, $a > b$; if he suspends 1 lb. mass from the arm b, and balances it by a quantity of tea suspended from the other arm, the mass of this tea is $\frac{b}{a}$ lb., so that he gains $1-\frac{b}{a}$ lb. In the next transaction if he suspends 1 lb. from a and balances it by tea, the mass of the tea is $\frac{a}{b}$ lb., so that he loses $\frac{a}{b}-1$. On the double transaction he *loses*, because $\frac{a}{b}-1$ is greater than $1-\frac{b}{a}$, the difference being $\frac{a}{b}+\frac{b}{a}-2$, which is positive, whatever the value of $\frac{a}{b}$ may be, because this difference is $\frac{(a-b)^2}{ab}$. This is the loss on a 2-lb. transaction, so that the loss per cent. is $50\frac{(a-b)^2}{ab}$.

7. If the weight of the balance and the scales is not negligible, must he gain or lose?

If the centre of gravity of the balance and scales is at the same side of the fulcrum as the shorter arm, if a known mass, p, is employed each time, and if the mass of the balance and scales is $> \dfrac{a-b}{x}p$, where x is the distance of the centre of gravity from the fulcrum, he will gain.

8. A nutcracker is 5 inches long; a nut is placed in it at a distance of 1 inch from the fulcrum, and the nut requires a pressure equal to the weight of 10 pounds to crack it; what effort must be used by the hand to crack the nut?

Ans. 2 pounds' weight.

9. A rectangular box uniformly loaded lies on the ground; its mass is 300 pounds; a crowbar 4 feet long whose mass is 10 pounds is inserted to a distance of 1 foot under the box; find the effort which must be applied to the crowbar to tilt the box.

Result. $42\frac{1}{2}$ pounds' weight.

10. On an axle 20 inches in diameter are mounted two wheels whose diameters are 5 and 4 feet; what mass can be raised over the axle by efforts of 40 and 30 pounds' weight applied by means of these wheels, respectively?

Ans. 192 pounds.

11. In a differential windlass in which the radii of the cylinders are 2 feet and $1\frac{1}{2}$ feet, and the radius of the circle described by the effort is 3 feet, what mass will be raised by an effort of 40 pounds' weight?

Ans. 480 pounds.

12. Through what vertical distance is the moveable pulley M (fig. 104) raised when the handle has completed a revolution, in the differential windlass?

Ans. $\pi(r-r')$.

13. In the arrangement of six pulleys represented in fig. 106 if W represents a basket in which a man of weight W is placed, and if the man wishes to raise himself by catching the free end of the rope and pulling, what force must he use?

Ans. $\dfrac{W}{7}$.

14. What force must he use if he raises himself by means of the system represented in fig. 107?

Ans. $\dfrac{W}{9}$.

15. If he raises himself by the system in fig. 108, what force must he use?

Ans. $\dfrac{W}{8}$.

16. In Questions 13, 14, 15, if the free end of a rope, before passing to the man's hand, is passed round a fixed pulley attached to the ground, what forces must he use to raise himself?

Ans. $\frac{W}{5}$, $\frac{W}{7}$, $\frac{W}{6}$, respectively.

17. If a man has to raise a given mass from the ground and has only one pulley for the purpose, what is the most advantageous method?

Ans. If he can fix the pulley to the top of the mass, and one end of the rope to the beam, passing the rope round the under surface of the pulley, let him pull the free end vertically up, and then the required tension is only $\frac{1}{2}W$. If he fixes the pulley to a fixed beam above the mass, attaching a rope to the body and passing this rope over the pulley, and if his own weight is less than that of the body, let him attach himself to the body and raise both together by pulling the free end of the rope. If his weight exceeds that of the body, it is better to pull the free end without attaching himself to the mass to be raised. In the former way he will have to exert a tension equal to $\frac{1}{2}(W+W')$, where W and W' are the weights of the given mass and of himself; and this tension is less than W.

GENERAL EXAMPLES OF EQUILIBRIUM

We now proceed to apply the two conditions (*a*) and (*b*) given on p. 164, to various cases of the equilibrium of bodies.

1. A bar, AB, fig. 111, whose centre of gravity, G, divides it into two segments AG, GB equal to a and b, rests with its ends on two vertical props of the same height; find the pressure on each prop.

Let N and S be the pressures exerted by the props on the bar; then, to find S, take moments about A, and we have $S \times AB - W \times AG = 0$—*i.e.*

$$S = W\frac{a}{a+b}.$$

Similarly $$N = W\frac{b}{a+b}.$$

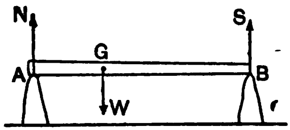


Fig. 111.

2. A plank 15 feet long, whose centre of gravity divides it into segments of 6 and 9, is placed with its ends resting on two vertical props of equal heights; where must a mass whose weight is one-third of that of the plank be placed on the plank so as to equalise the pressures on the props?

Ans. 3 feet from the end of the longer segment, GB (fig. 111).

3. A pole 28 feet long rests with its ends on two vertical props of the same height, on which it exerts pressures of 270 and 150 pounds' weight; find the position of the centre of gravity of the pole.

Result. It is 10 feet from one end.

4. A heavy cylindrical roller (fig. 112) whose centre of gravity, C, is its centre is to be pulled over an obstacle B by means of a horizontal pull, P, applied at C; find in terms of the weight, W, of the roller the required force.

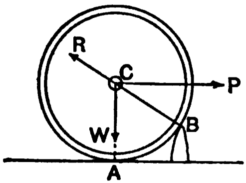


Fig. 112.

Let h be the height of the point of contact, B, of the roller with the obstacle above the ground, and r the radius of the roller. Then when the roller is about to roll over at B, it just comes out of contact with the ground at A. Hence it is in equilibrium (just bordering on motion) under the action of only three forces—viz. W, P, and the reaction at B. These three must meet in a point; and since P and W meet at C, the reaction at B must act through C—*i.e.* it is normal to the surface of the roller at B; and this is true whether the surfaces in contact at B are rough or smooth.

Now, by taking moments about B, we have

$$P(r-h)=W\sqrt{r^2-(r-h)^2},$$

$$\therefore P=W\frac{\sqrt{2rh-h^2}}{r-h},$$

which gives $P=\infty$ if $h=r$—an obvious result.

5. If fig. 112 represents a heavy circular wheel with spokes, and it is desired to drag it over the rough surface B by means of a rope attached to a spoke at a given point and pulled horizontally, will the initial motion of the wheel be a rolling over B or a slipping at B and A simultaneously?

Ans. If the line of the rope meets the vertical through C in a point D, the wheel will begin to roll over B provided that the angle CBD is less than the angle of friction between the wheel and the obstacle.

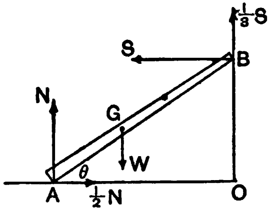


Fig. 113.

6. A ladder, AB, fig. 113, 15 feet long, whose centre of gravity, G, divides it into two segments of 6 and 9 feet, AG being 6, rests on the ground at A and against a vertical wall at B; the coefficients of friction against the ground and the wall are $\frac{1}{2}$ and $\frac{1}{3}$ respectively; find the position in which the ladder is just about to slip.

Let θ be the inclination of the ladder to the horizon when it is about to slip; let N and S be the normal pressures exerted on the ladder at A and B; then, since the end A is about to slip towards the left, the force of friction at A on the ladder is acting towards the right, and is $\frac{1}{2}N$. (See p. 69.) Similarly the friction at B on the ladder is $\frac{1}{3}S$, and it acts upwards. Let W be the weight of the ladder.

For equilibrium resolve the forces horizontally—*i.e.* express the fact that they have no horizontal component; thus:

$$S-\tfrac{1}{2}N=0 \quad . \quad . \quad . \quad (1)$$

Resolve them vertically, and we have

$$N+\tfrac{1}{3}S-W=0 \quad . \quad . \quad . \quad (2)$$

From these we have $S=\frac{3}{7}W$ and $N=\frac{6}{7}W$.

Finally, taking moments about A, we have

$$6\cos\theta . W-15\sin\theta . S-5\cos\theta . S=0 \quad . \quad . \quad . \quad (3)$$

or $$5(\cos\theta+3\sin\theta)S=6W\cos\theta,$$

or $$5(\cos\theta+3\sin\theta)=14\cos\theta$$

$$\therefore \tan\theta=\tfrac{3}{5}.$$

7. In the last, if the ladder is placed at an inclination $\tan^{-1}\frac{4}{5}$ to the ground, and a man whose weight is equal to that of the ladder ascends, how high can he go before the ladder slips?

Suppose that he reaches a point P whose distance from A is x, and let the notation be as in last example. Then, resolving horizontally and vertically, we have

$$S-\tfrac{1}{2}N=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$N+\tfrac{1}{3}S-2W=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which give $$S=\tfrac{6}{7}W \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking moments about A,

$$(6\cos\theta+x\cos\theta)W=S(15\sin\theta+5\cos\theta) \quad . \quad . \quad (4)$$

Substituting in (4) the value of S given by (3), and for $\tan\theta$ the value $\frac{4}{5}$, we have

$$x=\tfrac{60}{7}\text{ feet.}$$

8. In the last example, if a man is just able to reach the top of the ladder before it slips, what is his weight in terms of W, the weight of the ladder?

Ans. $\frac{1}{6}W$.

9. If the same ladder is placed at the inclination $\tan^{-1}\frac{2}{5}$ to the ground and has attached to it at A a cord which is pulled in the direction AO, find the limits between which the tension of this cord must lie so that the ladder may not move.

Result. $P_1=\frac{3}{22}W$, and $P_2=\frac{15}{2}W$, where P_1, and P_2 are the extreme values of the tension. When the tension is P_1, the ladder is on the point of slipping down, and when the tension is P_2, the ladder is on the point of slipping up the vertical plane.

10. If the ladder AB is 17 feet long, and $AG=6$ feet, while the coefficients of friction at A and B are $\frac{1}{2}$ and $\frac{4}{15}$ respectively; find the distance AO when the ladder is about to slip.

Result. $AO=15$ feet.

11. If the ladder in the last is placed so that AO is 8 feet, find, in terms of W (the weight of the ladder), the weight of the mass which must be suspended from B to cause the ladder just to slip down.

Result. $\frac{161}{15}W$.

12. AB (fig. 114) is a uniform trapdoor moveable round a fixed smooth horizontal axis at A; to B is attached a cord which, passing round a smooth pulley, C, in the horizontal line through A, such that $AC = AB$, sustains a mass of weight P, the weight of the door being W; find the position of equilibrium and the pressures on A and C.

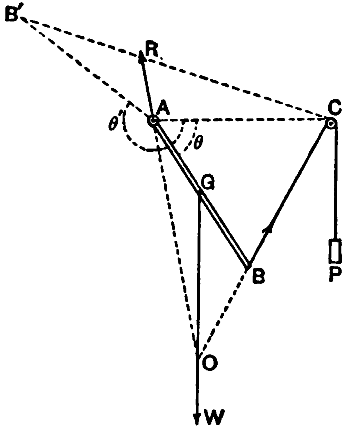


Fig. 114.

Let the angle CAB be θ, and let this define the position of equilibrium. Then there are three forces keeping AB in equilibrium — viz., its weight, W, the tension, P, of the cord BC, and the pressure at A. These must meet in a point; and if CB meets W in O, the reaction, R, of the axis must act in OA.

If $AB = 2a$, and the centre of gravity, G, of the door is the middle point, we have by taking moments about A for the equilibrium of the door,

$$P \times 2a \cos\frac{\theta}{2} - Wa\cos\theta = 0, \text{ or}$$

$$2W\cos^2\frac{\theta}{2} - 2P\cos\frac{\theta}{2} - W = 0, \quad . \quad . \quad . \quad (1)$$

a quadratic for $\cos\frac{\theta}{2}$. One value of $\cos\frac{\theta}{2}$ is negative, that is one value of θ is $>\pi$. This value, θ', corresponds to such a position of the door as AB', since it is impossible to exclude from the solution the supposition that the door is capable of revolving completely round the axis A. The position AB is given by the value

$$\cos\frac{\theta}{2} = \frac{P + \sqrt{P^2 + 2W^2}}{2W}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The force R produced by the axis is equal and opposite the resultant of P (in BC) and W; hence

$$R^2 = W^2 + P^2 - 2WP\cos\frac{\theta}{2}$$

$$= W^2 - P\sqrt{P^2 + 2W^2}, \text{ from (2).}$$

If S is the resultant pressure on the pulley C, we have

$$S^2=P^2+P^2+2P^2\cos\frac{\theta}{2}=2P^2\left(1+\cos\frac{\theta}{2}\right),$$

and S bisects the angle between BC and the vertical.

13. If in the last the mass of the trapdoor is 312 pounds, and the mass of P is 119, find the position of equilibrium.

Result. $\theta=\cos^{-1}\frac{119}{169}$.

14. If the axis at A is rough and fits tightly, so that there is friction all over its surface, it is no longer true (see p. 166) that the force which it exerts on the door passes through the centre of the axis. This force must, however, in all positions of equilibrium of the door pass through the point O in which the other two forces, P and W, acting on the door meet, and it is equal and opposite to the resultant of W and the tension, P, acting in OC.

Let the axis at A be rough, with contact all round, let the length AB be 2.6 feet, $W=48$ pounds, $P=52$ pounds, and suppose that the door takes a position of rest such that $BC=2$ feet; find the magnitude and line of action of the force produced on the door by the axis at A, no loss of tension taking place over the pulley at C.

Taking at O the total horizontal and vertical components, X and Y, of W and the tension, we have

$$X=52\sin\frac{\theta}{2}=52\times\tfrac{5}{13}=20 \text{ pounds' weight},$$

$$Y=48-52\cos\frac{\theta}{2}=48-52\times\tfrac{4}{13}=0.$$

The resultant of W and the tension is, therefore, a horizontal force acting to the right of O, and hence the resultant action of the axis on the door is a force of 20 pounds' weight acting in a horizontal line at O towards the left.

15. If in the last W is 68 pounds, while all the other data remain unaltered, find the resultant force exerted on the door by the axis.

Result. A force of $20\sqrt{2}$ pounds' weight acting at O in a line making $\frac{\pi}{4}$ with the vertical.

16. AB (fig. 115) is a uniform bar of weight W freely moveable round a horizontal axis at A; C is a smooth pulley fixed vertically above A; a cord attached to B passes over C and sustains a mass P; find the position of equilibrium.

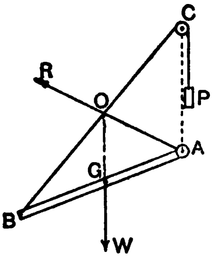


Fig. 115.

The bar is kept in equilibrium by three forces, W, the tension P acting in BC, and the reaction at A. These must meet in a point; hence if the vertical through G, the centre of gravity of the bar meets BC in O, R acts in AO.

Now we can employ the principle of the *triangle of forces*; for OCA

is a triangle whose sides are parallel to the three forces keeping AB in equilibrium; hence

$$\frac{OC}{CA}=\frac{P}{W} \quad \therefore \quad OC=\frac{P}{W}\,.\,CA.$$

Now since G is the middle of AB, $CB=2CO$, $\therefore$ $CB=2\frac{P}{W}\,.\,CA$, that is, CB is a known length, therefore B lies on the circumference of a given circle whose centre is C; but B also lies on a circle of radius AB and centre A; hence B is known, and therefore the position of equilibrium.

If $AB=2a$, and $AC=2b$, we shall find that

$$R^2=\tfrac{1}{2}\left(1+\frac{a^2}{b^2}\right)W^2-P^2.$$

One position of equilibrium, no matter what may be the values of P and W, is always a vertical position; and this may be the only one. It will be the only one if it is impossible to construct a triangle whose sides are a, b, and $2\frac{P}{W}b$.

17. BA (fig. 115) is a heavy bar of weight W, moveable round a fixed smooth horizontal axis at B, and resting at A against a vertical wall, AC; its centre of gravity, G, is $2\tfrac{1}{3}$ feet from B, the length of the bar is $5\tfrac{2}{3}$ feet, and B is 5 feet from the vertical wall; find the pressure at A, and the magnitude and line of action of the pressure at B.

If the bar is merely laid against the wall at A, and not *jammed* against it, there is no tendency to slip at A, so that, whether the wall is rough or smooth, no friction is called into play at A, and the reaction, S, of the wall is horizontal. The reaction at B passes through the centre of the axis at B and through the point in which S meets the vertical through G. The pressure S can be found by taking moments about B, and we have

$$S=\tfrac{105}{138}W,$$

while, since the reaction, R, at B is equal and opposite to the resultant of W and S, we have

$$R=1.27\,W, \text{ nearly.}$$

18. AB and BC, fig. 116, are two uniform planks the first resting on a

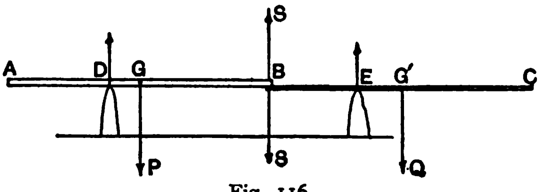


Fig. 116.

prop at D and the second on a prop at E, and just overlapping at B; the

props are of the same height; find the position of equilibrium, assuming all necessary data.

Let $AB=2a$, $BC=2b$, weight of $AB=P$, weight of $BC=Q$, and let the distance between the props be c. Then if S is the pressure of each plank on the other at B, we shall obtain a value of S in terms of P by considering the separate equilibrium of AB and taking moments about D; and another value of S, in terms of Q, by considering the equilibrium of BC and taking moments about E. Let $DB=x$; then x defines the position of equilibrium.

By moments about D for the equilibrium of AB, we have

$$S.x-P(x-a)=0, \therefore S=\frac{x-a}{x}.P \quad . \quad . \quad . \quad (1)$$

Again, $BE=c-x$, and $EG'=b-c+x$, where G' is the middle point of BC; therefore by moments about E for the equilibrium of BC,

$$S(c-x)-Q(b-c+x)=0, \therefore S=\frac{b-c+x}{c-x}.Q \quad . \quad (2)$$

Equating the values of S in (1) and (2), we have

$$(P+Q)x^2+\{(b-c)Q-(a+c)P\}x+acP=0, \quad . \quad (3)$$

a quadratic for x, which shows that there are two positions, if any.

Thus, if $P=4Q$, $AB=12$ feet, $BC=8$, and $DE=10$, there is one position in which the points G and G' coincide with D and E; and another in which $DB=8$ feet.

19. A body of any shape rests with a flat base on a rough horizontal

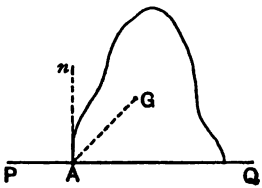


Fig. 117.

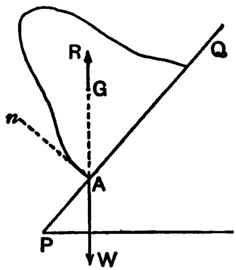


Fig. 118.

plane which is gradually tilted up; find whether the body will begin to slide or to tumble.

Let fig. 117 represent the body on the horizontal plane PQ. Then if PQ is gradually tilted up, as in figs. 48, 49, 50, p. 66, and the body still rests, the reaction, R, of the plane must always be equal and opposite to W, the weight of the body. Now if we can reach such a position as that in fig. 118, in which W

passes through the extreme point A of the base of the body, the whole reaction of the plane acts at A—which means that the base of the body is just out of contact with the plane PQ except at the single point A—*i.e.* that the body is just about to tilt round A. But in order that the vertical line AG (where G is the centre of gravity of the body) should be a possible line of action of R, the angle nAG must be less than the angle of friction between the body and the plane, An being the normal to PQ at A. Hence, if λ is this angle of friction, we see in fig. 117 without tilting the plane at all, that tilting round A will take place if

$$\angle\, nAG \text{ is less than } \lambda;$$

and, of course, if $\angle\, nAG$ is $>\lambda$, the position in fig. 118 cannot be reached, so that sliding must have taken place.

Hence if μ is the coefficient of friction,

tumbling over will take place if $\tan nAG < \mu$,
and sliding " " $\tan nAG > \mu$.

20. A uniform solid cone whose height is 12 feet and diameter of base 4 feet is placed with its base on a rough horizontal plane, the coefficient of friction being $\frac{1}{2}$; if the plane is gradually tilted up, will the cone slide or tumble?

(The centre of gravity of a solid homogeneous cone of height h is at a distance $\dfrac{h}{4}$ from the base.)

Ans. It will slide.
(Here $\tan nAG = \frac{2}{3}$.)

21. In the last question if the diameter of the base is still 4 feet, what is the least height of the cone that will allow of tumbling?

Ans. 16 feet.

22. A thick uniform board in the shape of an isosceles triangle whose height is 16 inches and base 6 inches long is placed vertically with its base on a rough horizontal plane for which the coefficient of friction is $\frac{3}{4}$; if the plane is gradually tilted up, will the motion of the board be sliding or tumbling?

Ans. Tumbling. (The centre of gravity of a triangular area of height h is $\dfrac{h}{3}$ from the base.)

23. A brick whose edges are 12, 6, and 4 inches long is placed with one face on a rough horizontal plane for which the coefficient of friction is $\frac{5}{12}$; can it be placed on the plane so that when the plane is gradually tilted up, the motion will be tumbling?

Ans. Yes.

24. If the brick is placed with the sides 12 and 4 on the plane, at what inclination of the plane will motion take place?

Ans. $\tan^{-1} \frac{5}{12}$.

25. Two pegs, A and B, are driven into a vertical wall, the distance between them being given, and the inclination of the line AB to the horizon being $\tan^{-1} \frac{5}{12}$; a heavy rod is laid on the pegs, the coefficients of friction

at the pegs being $\frac{1}{2}$ and $\frac{1}{3}$; find the position of the rod when it is just about to slip down, and the corresponding normal pressures on the pegs.

Result. The centre of gravity of the rod is midway between the pegs, and the pressures are each equal to $\frac{6}{13}W$, where W is the weight of the rod.

26. Two long thin pegs, A and B, are driven into a vertical wall, and they project outwards from it, the line AB being vertical; a rod CD of weight W passes down along the line AB having the upper peg, B, to the right and the lower to the left of the rod; the lower extremity, D, of the rod rests on a fixed inclined plane whose inclination to the ground is i; calculate the pressure on this plane and also the pressures on the pegs.

Result. Let the distance BA be c, let AD be a, and let the pressures at D, A, and B be N, R, and S; then $N = W \sec i$;

$$R = W\frac{a+c}{c}\tan i;\ S = W\frac{a}{c}\tan i.$$

27. Two uniform ladders, BA and CA, of equal lengths and weights, are freely jointed together at one common extremity, A, while the other extremities, B, C, rest on a rough horizontal plane, the plane BAC being vertical; find the value of the angle BAC when the ends B and C are just about to slip.

Result. If μ is the coefficient of friction between the ladders and the ground, $\tan\frac{BAC}{2} = 2\mu$. (The reaction at A is horizontal.)

EXAMINATION ON CHAPTER X

1. State the conditions that must be satisfied by any system of co-planar forces acting on a rigid body in order that the body may be in equilibrium.

Can these conditions be reduced to *one*?

2. What is the condition for the equilibrium of *two* forces?

Give the conditions for *three* forces.

3. How much is always certainly known concerning the resultant action of a smooth cylindrical axis on a body moveable round it?

If the axis is rough and contact exists all the way round, do we know anything about the resultant action?

4. Can the true weight of a body be found by means of a false balance?

5. Give any common instances of various applications of the lever.

What is the relation between the *effort* and the *resistance* in every lever?

6. What is meant by the *mechanical advantage* of any machine?

7. What is the objection to multiplying very greatly the mechanical advantage of a differential windlass.

8. Sketch a train of toothed wheels, and say which are *drivers* and which *followers*.

9. If a tradesman uses a false balance, and puts the substance to be weighed as often into one pan as into the other, does he gain or lose?

10. If a body of given shape the position of whose centre of gravity is known is placed with a flat base in contact with a rough plane which is gradually tilted up, how can you tell at once whether the body will slide or tumble as the plane is tilted?

CHAPTER XI

CENTRES OF PARALLEL FORCES AND OF GRAVITY

50. **Geometrical Theorem.**—A and B (fig. 119) are any two points whose distances AM and BN from any plane, MN, are z_1 and z_2; G is a point on AB dividing AB into two segments such that $AG : GB = m : n$, and GP, or z, is the distance of G from the plane MN; then

$$z = \frac{nz_1 + mz_2}{m+n}. \qquad . \quad . \quad . \quad . \quad . \quad (a)$$

To prove this draw the line Apn parallel to MPN; then $z = Gp + pP = Gp + z_1$. But $\frac{Gp}{Bn} = \frac{AG}{AB} = \frac{m}{m+n}$; $\therefore$ $Gp = \frac{m}{m+n}(z_2 - z_1)$, since $Bn = z_2 - z_1$. Hence

$$z = \frac{m}{m+n}(z_2 - z_1) + z_1$$

$$= \frac{nz_1 + mz_2}{m+n},$$

which was to be proved.

In particular, if G is the middle point of AB, $z = \frac{1}{2}(z_1 + z_2)$.

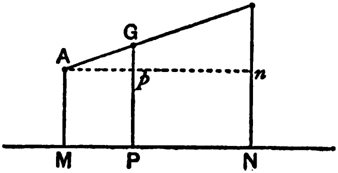


Fig. 119.

Note.—The lines AM, BN, GP may be *oblique* to the line MN, provided that they are parallel, and this result will still hold.

If at A and B act two like parallel forces of magnitudes P_1 and P_2, having any common direction whatever, their resultant is $P_1 + P_2$, and its line of action is found (see p. 152) by dividing AB at a point, G, such that $AG : GB = P_2 : P_1$; then the resultant acts

through G in a direction parallel to that of P_1 and P_2. By the above the distance, z, of G from any plane from which the distances of A and B are z_1 and z_2, is given by the equation

$$z=\frac{P_1z_1+P_2z_2}{P_1+P_2}. \quad . \quad . \quad . \quad . \quad (1)$$

This is true no matter what the common direction of P_1 and P_2 is; so that if P_1 and P_2 keep turning round the fixed points A and B while remaining always parallel to each other, their resultant keeps turning round the fixed point G.

This point G is called the *centre* of the two parallel forces, so that we have the definition: *the centre of two parallel forces is that fixed point through which their resultant always passes if each of the two given parallel forces turns round a fixed point.*

Now let there be any number of parallel forces, P_1, P_2, P_3, P_4, ... (fig. 120), each acting at a *fixed* point, these points being A_1, A_2, A_3, A_4, ... respectively, and each turning round its point of application, the system always remaining a parallel set; then their resultant will always pass through a certain *fixed* point, which is called the *centre* of the given parallel forces.

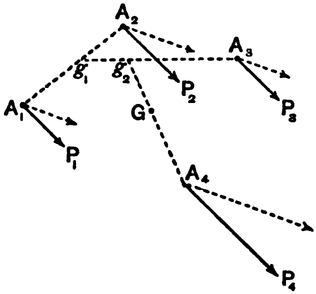


Fig. 120.

For, the resultant of P_1 and P_2 acts through the point g_1 on A_1A_2 such that $A_1g_1 : A_2g_1 = P_2 : P_1$ and turns round g_1 as P_1 and P_2 turn round A_1 and A_2. This resultant is P_1+P_2; and the resultant of P_1, P_2, and P_3 is found by joining g_1 to A_3 and taking the point g_2 on g_1A_3 such that $g_1g_2 : A_3g_2 = P_3 : P_1+P_2$. This resultant is $P_1+P_2+P_3$, and it will keep turning round g_2 when P_1, P_2, and P_3 turn round A_1, A_2, A_3 (while remaining parallel, of course).

Similarly, the resultant of P_1, P_2, P_3 and P_4 is found by joining g_2 to A_4 and taking on g_2A_4 the point G such that $g_2G : A_4G = P_4 : P_1+P_2+P_3$. This resultant is $P_1+P_2+P_3+P_4$, and it continuously turns round G if the individual forces keep turning round the fixed points A_1, A_2, A_3, A_4.

The various points g_1, g_2, G are all *fixed*, because their positions depend on the *magnitudes* or P_1, P_2, P_3, P_4, and not at all on their common direction.

Moreover, the distance of G from any plane is very simply expressed in terms of the distances of the fixed points A_1, A_2, . . . from this plane.

For, let the distances of A_1, A_2, A_3, A_4 from the plane be z_1, z_2, z_3, z_4; and let the distances of g_1, g_2, G from this plane be p_1, p_2, $\bar{z}$. Then by (1) we have

$$p_1=\frac{P_1z_1+P_2z_2}{P_1+P_2}. \qquad (2)$$

Again, since $g_1g_2:g_2A_3=P_3:P_1+P_2$, we have by (1)

$$p_2=\frac{(P_1+P_2)p_1+P_3z_3}{P_1+P_2+P_3}$$

$$=\frac{P_1z_1+P_2z_2+P_3z_3}{P_1+P_2+P_3}, \text{ by (2)} \qquad (3)$$

Similarly

$$\bar{z}=\frac{(P_1+P_2+P_3)p_2+P_4z_4}{P_1+P_2+P_3+P_4}$$

$$=\frac{P_1z_1+P_2z_2+P_3z_3+P_4z_4}{P_1+P_2+P_3+P_4}, \text{ by (3)} \qquad (4)$$

If the forces P_1, P_2, . . . are not all in the same sense, the formula for $\bar{z}$ still holds: we have merely to mark all forces having one common sense positive and those having the opposite sense negative. For, returning to fig. 119, if P_2 and P_1 act at B and A in opposite senses, the point G (see fig. 87, p. 153) will be on BA produced either through A or through B according as P_1 is $>$ or $<$ P_2. Suppose $P_1>P_2$; then G will be to the left of A in fig. 119, and since $GA:GB=P_2:P_1$, it is easy to see by exactly the same proof as before that the distance, z, of G from the plane, MN, of reference is given by the equation

$$z=\frac{P_1z_1-P_2z_2}{P_1-P_2}; \qquad (5)$$

and hence, if the forces in fig. 120 at A_1, A_2, A_3, A_4 were P_1, $-P_2$, $-P_3$, P_4, equation (4) would become

$$\bar{z}=\frac{P_1z_1-P_2z_2-P_3z_3+P_4z_4}{P_1-P_2-P_3+P_4}. \qquad (6)$$

Thus when some of the parallel forces act in one sense and some in the opposite, there will be positive and negative forces involved in our formula for $\bar{z}$. But also some of the distances z_1, z_2, z_3, z_4 may be positive and some negative. For example, take such a figure as fig. 121, in which *MN* is the reference plane — *i.e.* the plane from which the distances of A_1, A_2 . . . are measured. Let the forces be taken as 6, 8, − 12, − 10 acting at A_1, A_2, A_3, A_4; then if distances measured from the upper side of the plane *MN* are called positive, distances below it must be called negative. Let, then, the distances of the above points from *MN* be, respectively, 4, − 6, 10, − 8, and let it be required to find the distance from *MN* of the *centre* of the given parallel forces.

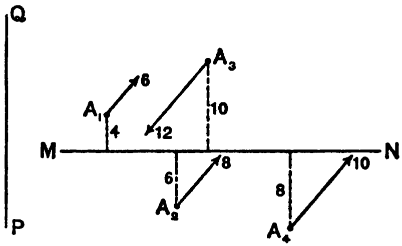


Fig. 121.

The most convenient way of employing the formula (4) or (6) is exhibited in the following table:

Forces.	Distances from *MN*.	Products.
6	4	24
8	− 6	− 48
− 12	10	− 120
− 10	− 8	80
− 8		− 64

. . (β)

which consists of a column of forces, a column of distances of their points of application from the given reference plane, and a column of products, each term in this third column being the product of the force in the first column and the corresponding distance in the second.

Then the required distance of the centre from *MN* is

obtained by dividing the sum, -64, of the third column by the sum, -8, of the first. This gives

$$z=\frac{-64}{-8}=8.$$

Hence the resultant is a force of 8 units acting in the negative sense, and the centre, G, of the system is 8 units of length from MN at its upper side.

To determine completely the position of G, we must find its distance from some other reference plane, such as PQ, from which the distances of A_1, A_2, . . . must be given; and to the above table we should add a fourth column headed "distances from PQ," and a fifth headed "products." The sum of the fifth column divided by -8, the sum of the first gives the distance of G from PQ.

The result expressed by equation (4) or (6), or by the table (β), for finding the position of the centre of a system of parallel forces we shall call the principle of *Plane-Moments*, since each force is multiplied by the distance of its point of application from a *plane*.

[*Note.*—The student will observe that this principle of Plane-Moments holds for *parallel* forces only—not for a system of non-parallel forces, whose resultant must be found by the method explained on p. 160.]

Centre of gravity of two particles.—Supposing that there is a very small particle of weight w at A (fig. 119) and another of weight w' at B, since the lines of action of these weights are both vertical, the forces w and w' are two like parallel forces; and if the particles are rigidly connected together we may consider the *resultant* weight, $w+w'$, and its point, G, of application. This point is on AB, and is such that

$$\frac{AG}{GB}=\frac{w'}{w}. \quad . \quad . \quad . \quad . \quad (7)$$

The point G is called the *centre of gravity* of the two particles.

Since the weights w, w', no matter in what units they may be measured, are proportional to masses of the particles, the point G may also be described as the *centre of mass* of the particles. It is a matter of indifference which term we use.

Centre of gravity, or centre of mass, of any system of particles.—Supposing that at any points, A_1, A_2, A_3, A_4, . . . (fig. 120)

there are very small particles of weights $w_1, w_2, w_3, w_4, \ldots$ the centre of gravity of the first two is at the point g_1 such that $A_1g_1 : g_1A_2 = w_2 : w_1$; and these two can be imagined to be replaced by a single particle of weight $w_1 + w_2$ at g_1. Again, the centre of gravity, g_2, of this particle and w_3 is found by taking g_2 on g_1A_3 such that $g_1g_2 : g_2A_3 = w_3 : w_1 + w_2$, and finally, by taking G on g_2A_4 such that $g_2G : GA_4 = w_4 : w_1 + w_2 + w_3$, the point G thus obtained is called the centre of gravity, or centre of mass, of the system of particles.

All this holds, no matter how numerous or how close together the system of particles may be; so that they may form a continuous solid body.

From what has been explained, the position of the final point G depends merely on the magnitudes of the weights (and therefore of the *masses*) of the individual particles, $w_1, w_2, \ldots$ and not on the direction of the system of parallel forces formed by these weights.

Though we cannot alter the common direction of these weights by moving the whole Earth (to whose attraction on the particles they are due) to the right or left of the given body, we can produce precisely the same effect by merely turning the body itself round into different positions; and then, no matter into what position it is turned, the resultant, $w_1 + w_2 + w_3 + \ldots$. of the weights of the particles continues to pass through the *same point*, G, in the body. This point is called the centre of gravity, or centre of mass, of the body. Formally, then, we define the centre of gravity of any body as follows:

> *The centre of gravity of any body is that point through which the resultant of the weights of all the indefinitely-small particles into which the body may be conceived as divided continues to pass no matter what position the body may occupy at the surface of the Earth.*

If the body is divided into particles whose weights are $w_1, w_2, w_3, \ldots$ at points whose distances from any plane of reference are $z_1, z_2, z_3, \ldots$ the distance, $\bar{z}$, of the centre of gravity of the body from the plane is given by the equation

$$\bar{z} = \frac{w_1z_1 + w_2z_2 + w_3z_3 + \ldots}{W}, \qquad (8)$$

where W is put for $w_1 + w_2 + w_3 + \ldots$, the weight of the whole body.

Another definition of the centre of gravity, or centre of mass, of a body follows at once from (8), viz.:

The centre of gravity of any body, or system of particles, is that point whose distance from any plane is the mean distance of the body, or system of particles, from the plane.

51. **Special Cases of Centres of Gravity or Centres of Mass.**—*Uniform Triangular Plate.* Let ABC, fig. 122, be a triangular plate of uniform, and very small, thickness. [Any thin plate is called a *lamina.*] Now this plate may be broken up into an indefinitely great number of narrow strips, such as $PQRS$, by drawing a series of very close lines PQ, RS, . . . parallel to any side, AB, of the triangle. The centre of gravity of the strip $PQRS$ is its middle point, since the strip is of uniform thickness; and the centres of gravity of all the other strips are also their middle points.

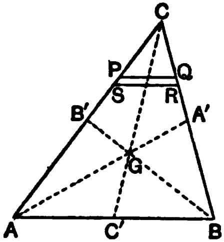


Fig. 122.

Now all these middle points lie on the line joining C to the middle point, C', of AB; therefore the centre of gravity of the whole plate lies somewhere on the line CC'.

Similarly, since the plate could be broken up into strips parallel to another side, BC, the centre of gravity of the plate must lie on AA', where A' is the middle point of BC.

Hence we conclude two things—

(*a*) The centre of gravity must be the point of intersection, G, of the two bisectors AA' and CC' of the sides BC and AB; and

(*b*) the bisector BB' of the third side AC drawn from B must pass through G; that is, *the three bisectors of the sides of any triangle drawn from the opposite vertices must meet in a point*—which is the centre of gravity of the triangle.

This is a well-known result in Geometry, and our proof here rests on the principle that a body has only one centre of gravity.

Moreover, it is a well-known geometrical theorem that the point G is a point of trisection of each of the bisectors AA', BB', CC'; that is, $C'G = \frac{1}{3}C'C$, etc.; and this result we deduce

dynamically thus: at the points A, B, C suppose three equal particles each of mass m placed, and let us find their centre of mass. Obviously C' is the centre of mass of m at A and m at B, so that these particles can be replaced by $2m$ at C'. Then, drawing $C'C$ and taking a point, G, such that $C'G:GC = m:2m = 1:2$, we can replace the three particles by one of mass $3m$ at G, which is the centre of gravity of the three equal particles. Now this point is the same as the centre of gravity of the uniform triangular plate ABC, because the centre of gravity of the three particles lies on $C'C$; and for a similar reason it must lie on $A'A$; that is, the construction for it is the same as that for the centre of gravity of the plate.

Of course since $C'G:GC=1:2$, we have

$$C'G = \tfrac{1}{3}C'C.$$

Similarly $A'G = \frac{1}{3}A'A$, etc.

We may speak of G as the centre of gravity of the triangular *area* ABC, instead of the *uniform thin plate* ABC; and, indeed, the former is the usual manner of speaking.

Again, for the purpose of finding the position of its centre of gravity, we may replace the triangular lamina or area by three equal particles placed *at the middle points of the sides* instead of at the vertices.

For if m, m, m are placed at A', B', C', the centre of gravity of the particles at A' and B' is the mid point of $A'B'$; but this lies on $C'C$, therefore the centre of gravity of the three particles lies on $C'C$; and similarly it lies on $B'B$, and on $A'A$; therefore it is the same as the centre of gravity of the triangle.

Cor. 1.—The *perpendicular* distance of the centre of gravity of a triangle from any side AB is $\frac{1}{3}h$, where h is the perpen dicular from C on AB.

Cor. 2.—The distance of the centre of gravity of a triangle from any plane is $\frac{1}{3}$ of the sum of the distances of the vertices, or of the middle points of the sides, from that plane.

For, if z_1, z_2, z_3 are the distances of A, B, C from the plane and if three particles each of mass m are placed at these points equation (8), p. 193, gives

$$\bar{z} = \frac{mz_1 + mz_2 + mz_3}{3m}$$
$$= \tfrac{1}{3}(z_1 + z_2 + z_3)$$

for the distance of their centre of gravity from the plane.

(Note that no such result holds for a quadrilateral or any other area than a triangular one.)

So far as the position of the centre of gravity of a triangular plate is concerned, the three equal particles which may replace it, whether at A, B, C or at A', B', C', may have any common mass, m, whatever; but to be completely equivalent to the given plate the three masses should be each equal to $\frac{1}{3}$ of the mass of the plate; and this becomes necessary when we have a plate in the form of a polygon of any number of sides, which we consider as formed by a number of different triangles.

Centre of Gravity of a Trapezium.—Let $ABCD$ (fig. 123) be a uniform thin lamina in the form of a trapezium, the parallel sides, AB and CD, having lengths $2a$, $2b$, and the perpendicular distance between them being h.

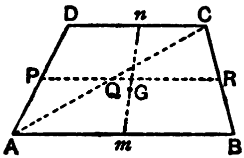


Fig. 123.

If m and n are the mid points of AB and CD, it is well known by elementary geometry that mn bisects all lines drawn across the area parallel to AB. Hence, since the area could be broken up into narrow strips parallel to AB, and since the centre of gravity of each strip is its middle point, it is clear that the centre of gravity of the whole trapezium must lie somewhere on mn.

Let us find its distance from AB. For this purpose we can consider the trapezium as consisting of the two triangles ACB and ACD; their areas are ah and bh, so that they may be represented by a and b, since it is with their *ratios* only that we are concerned. Also the perpendicular distances of their respective centres of gravity from AB are $\frac{1}{3}h$ and $\frac{2}{3}h$; hence, forming a table such as (β), p. 191, we have:

Areas.	Distances from AB.	Products.
a	$\frac{1}{3}h$	$\frac{1}{3}ah$
b	$\frac{2}{3}h$	$\frac{2}{3}bh$

The sum of the third column $=(a+2b)\dfrac{h}{3}$, and that of the first $=a+b$; hence by the rule of p. 192

$$\bar{z} = \frac{a+2b}{a+b} \cdot \frac{h}{3},$$

where $\bar{z}$ is the distance of the centre of gravity, G, from AB.

Of course the point G can be found at once by joining the centre of gravity, g, of ACB to that, g', of ACD; then G is the point of intersection of mn with gg'.

Owing to the great utility of the *particle method* of finding centres of gravity of uniform laminæ, or areas, bounded by right lines, we give yet another illustration of it for finding the centre of gravity of this trapezium.

Dividing the area into the two triangles ACB and ACD, whose areas are proportional to a and b, replace each by three particles at the *middle points* of the sides. If the mid points of AD, AC, CB are P, Q, R, these points lie on a right line parallel to AB. Then, replacing the area ACD, we have particles b, b, b, at P, Q, and n; similarly, replacing ACB, we have particles a, a, a, at Q, R, m.

Hence at P, Q, R, m, n we have particles whose masses are proportional, respectively, to b, $a+b$, a, a, and b. Now calculate the distance of G from the line PQR which contains three of the five particles, and form the table:

Masses.	Distances from PR.	Products.
b	$-\frac{h}{2}$	$-\frac{1}{2}bh$
b	0	0
$a+b$	0	0
a	0	0
a	$\frac{h}{2}$	$\frac{1}{2}ah$
$3(a+b)$		$\frac{1}{2}(a-b)h$

in which we have taken the lower side of PR as the positive side, so that the distance of the particle a at m from PR is $\frac{h}{2}$, while that of the particle b at n is $-\frac{h}{2}$.

Hence the distance of G from PR is

$$\frac{a-b}{a+b} \cdot \frac{h}{6},$$

which is easily seen to agree with the previous result.

The advantage of taking PR as the reference line (or rather the *plane* through PR perpendicular to the plane of the figure as the reference plane) is obvious, owing to the number of zero products which it gives in the third column of the above table.

The student will find this method and this particular selection of a reference plane of very great use in hydrostatical calculations.

Solid Homogeneous Pyramid.—Let ABC (fig. 124) be any triangle, suppose in the plane of the paper, and D any point above this plane; then, joining D to A, B, and C, we obtain the figure of a triangular pyramid, the volume of which we shall imagine to be filled with some homogeneous substance. We propose to find the centre of gravity of the body so formed. Let S be the centre of gravity of the area ABC; draw DS. Then if we cut the pyramid across by any plane, PQR, parallel to the base ABC, the line DS will cut the area PQR in a point s which is the centre of gravity of this area; in other words, if we imagine the pyramid as sliced into a very great number of very thin triangular plates by a series of very close planes, such as PQR and pqr, all parallel to the base ABC, the line DS will pierce each of these plates at its centre of gravity. This is sufficiently obvious from the fact that the solid figure $ABCD$ is exactly similar to the solid figure $PQRD$, and the triangle ABC to the triangle PQR, so that if any line through D cuts the area ABC in a point related in any

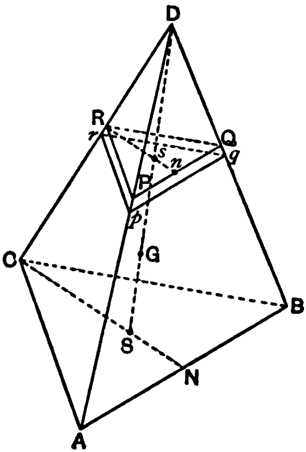


Fig. 124.

particular way to the area ABC, it must cut the area PQR in a point related in precisely the same way to the area PQR. But, for the sake of formality, we give a special proof that s is the centre of gravity of PQR. The lines Rs and CS, since they lie in the parallel planes PQR and ABC, can never meet; but these lines lie also in the *same* plane, CDS; therefore they are parallel lines. Let CS meet AB in N, and Rs meet PQ in n. Then the sides CA, AN, NC of the triangle CAN are parallel to the sides RP, Pn, nR of the triangle RPn; these triangles are therefore similar; hence $Pn : nR = AN : NC$.

Similarly, the triangles CBN and RQn are similar, therefore $Qn : nR = BN : NC$; hence

$$\frac{Pn}{nQ} = \frac{AN}{NB} = 1,$$

because, since S is the centre of gravity of ABC, $AN = NB$. Therefore n is the mid point of PQ. In the same way Qs produced bisects RP; that is, s is the centre of gravity of RPQ.

Now since the centres of gravity of all the thin plates lie on DS, the centre of gravity of the pyramid must lie on DS.

Similarly, by joining B to the centre of gravity of the triangle ADC, the centre of gravity lies on the joining line.

Now this is exactly the result which we obtain if we seek the centre of gravity of four equal particles each of mass m placed at the vertices $ABCD$, since S is also the centre of gravity of the three equal particles at A, B, C.

Consequently *the centre of gravity of the pyramid coincides with that of four equal particles placed at its vertices*; and the theorem of Plane-Moments (p. 192) shows that the distance of this point from the plane ABC is $\frac{h}{4}$, if h is the perpendicular from D on the plane ABC; that is, the centre of gravity, G, of the pyramid lies on SD and is one-quarter of the way up this line, or

$$SG = \tfrac{1}{4} SD.$$

Cor.—The distance of the centre of gravity of a triangular pyramid from any plane is $\frac{1}{4}$ of the sum of the distances of its vertices from that plane.

Right Circular Cone.—Let $ABCD \ldots H$ (fig. 125) be a circle in the plane of the paper, S its centre, and SO the perpendicular through S to the plane of the circle, O being any point on this perpendicular. If we draw lines, OA, OB, OC, ... from O to all points on the circle we form a figure called a *right circular cone.* Let the space inside this cone be filled with any homogeneous substance, and let it be required to find the centre of gravity of the body thus formed. By dividing the cone up into an indefinitely great number of very thin circular plates by planes all parallel to the base $ABCD \ldots H$, since the line OS cuts each at its centre (which is its centre of gravity), we see that the centre of gravity of the whole cone lies somewhere on SO.

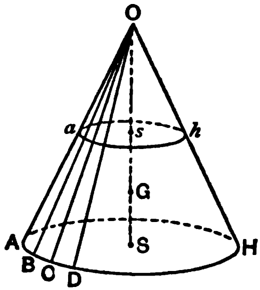


Fig. 125.

Now let the circle $ABCD \ldots H$ be divided into an indefinitely great number of small lengths, AB, BC, CD, ... so small that each may be considered to be a straight line, and it is clear that we can consider the cone as composed of a number of very thin *triangular pyramids*, $ABCO$, $ACDO$, ... all having a common vertex at O. By what precedes the centre of gravity of each of these pyramids is at a height $\frac{1}{4}SO$ from the plane of the base (not, of course, on SO, but on the line joining O to the centre of gravity of its triangular base ABC, or ACD, etc.). Hence the centres of gravity of all the triangular pyramids lie in a plane parallel to the base at a height $\frac{1}{4}SO$ above it; therefore the centre of gravity of the whole cone lies in this plane, and therefore if G is this point

$$SG = \tfrac{1}{4}SO.$$

This result holds for any kind of solid homogeneous cone, whether it stands on a circular base or not. If we describe any closed curve in the plane of the paper, and take any point whatever, O, above the plane and from O draw lines to all the points of the curve, we get a cone. If this is filled with any homogeneous substance, the centre of gravity of the body thus formed is obtained by joining O to the

centre of gravity, S, of the base curve and taking on the line SO a point G such that $G = \frac{1}{4}SO$.

This is obvious from the fact that all sections of the cone made by planes parallel to the base are similar curves, and therefore the line OS pierces each of them in its centre of gravity. Also, as above, we can consider the cone as made up of an indefinitely great number of thin triangular pyramids whose centres of gravity all lie in a plane parallel to the base at a height equal to $\frac{1}{4}$ of the perpendicular height of the cone.

Curved Surface of a Cone.—If a thin sheet of uniform thickness, such as a sheet of paper, is formed into the surface of a cone, we may speak of the centre of gravity of this mass as the centre of gravity of the curved surface. Now by drawing from the vertex a series of very close lines down along the surface, each adjacent pair of these lines and the small element of the circular base which they intercept may be considered as forming a slender triangle, whose centre of gravity is $\frac{1}{3}$ of the way up the line joining the vertex to the mid point of the base. Hence the centres of gravity of all the slender triangles of which the curved surface is made up lie on a plane parallel to the base at a distance from the latter equal to $\frac{h}{3}$, where h is the height of the cone.

When a body of any shape rests on a horizontal plane the contact taking place at any number of points (which may form a continuous curve) the condition that it shall not topple over is that the vertical line drawn through its centre of gravity falls inside the area of some polygon which can be formed by joining the points of contact by right lines. This is obvious, because if the body is in equilibrium under the action of its own weight and pressures exerted by the plane at the points of contact, the resultant of these pressures must be equal and opposite to the weight. This result is usually expressed by saying that *the vertical through the centre of gravity must fall within the base.*

For example, if we take a uniform solid cylinder—say a wooden rod—and cut it across by two parallel planes AB, CD (fig. 126) obliquely to its length, and then place it on

a horizontal table with one of its plane bases on the table, it will topple over unless its length is less than a certain quantity. If S and O are the centres of its faces, its centre of gravity, G, is mid-way between S and O; and for equilibrium the point P in which the vertical through G cuts the base must be inside the area AB. If the length SO were such that P fell at B, the body would be on the point of toppling round B.

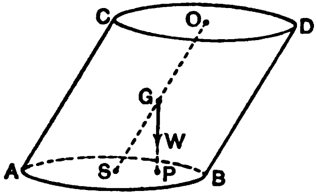


Fig. 126.

EXAMPLES

1. From a circular area, C (fig. 127) of cardboard of uniform thickness is punched out a circular area D; find the centre of gravity of the remainder.

Let A and B be the centres of the circles C and D; let the radii of these circles be, respectively, a and b, and let $AB=c$. Then considering the whole area C as being made up of D and the remainder, we see that the centre of gravity, G, of the remainder is on BA produced through A. Take Plane-Moments about the plane through A perpendicular to AB. Then since the distance of the centre of gravity, A, of the resultant area C from this plane is zero, we have

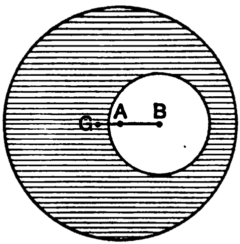


Fig. 127.

$$\pi(a^2-b^2).AG-\pi b^2.AB=0,$$

$$\therefore\ AG=\frac{b^2}{a^2-b^2}.c.$$

2. If C (fig. 127) is a sphere in which there is a spherical cavity D, find the position of G.

Result. $AG=\dfrac{b^3}{a^3-b^3}.c.$

3. $AHha$ (fig. 125) is a homogeneous frustum of a right cone, the radii of whose circular bases AB and CD are, respectively, 5 feet and 3 feet, the thickness, Ss, of the frustum being 6 feet; find the position of its centre of gravity.

Complete the cone, and let its vertex be O. Then we have $\dfrac{SO}{Os}=\frac{5}{3}$, and $Ss=6$, $\therefore\ SO=15$, $sO=9$. Now take plane-moments about the base $ABCH$, and equate the moment of the

whole cone to the sum of the moments of the small cone aOh and the frustum. Let V=volume of whole cone; then, since the small one is similar to the whole cone, and the volumes of similar solids are proportional to the *cubes* of corresponding linear dimensions, the volume of the small cone $= V(\frac{3}{5})^3 = \frac{27}{125}V$, and therefore the volume of the frustum $= \frac{98}{125}V$. Hence vol. of whole cone : vol. of small cone : vol. of frustum $= 125 : 27 : 98$. Let z be the distance of the centre of gravity of the frustum from the base AB. The distance of the centre of gravity of the whole cone from $ABCH$ is $\frac{15}{4}$, and the distance of the centre of gravity of the small cone is $\frac{9}{4}+6$; hence we have the table

Volumes.	Distances from AB.	Products.
27	$\frac{33}{4}$	$\frac{891}{4}$
98	z	$98z$
125		$\frac{891}{4}+98z$

Dividing the sum of the third by the sum of the second column we get the distance of the centre of gravity of the whole cone; but this is $\frac{15}{4}$,

$$\therefore \ \frac{15}{4} = \frac{891+392z}{4 \times 125}$$

$$\therefore \ z = 2\tfrac{25}{49}.$$

4. The radii of the bases of a homogeneous frustum of a right cone are 8 feet and 4 feet, and the thickness of the frustum is 12 feet; find the position of its centre of gravity.

Result. It is $4\frac{5}{7}$ feet from the larger base.

5. The radii of the bases of a conical frustum are 8 and 4 feet; what must its thickness be so that its centre of gravity may be $2\frac{3}{4}$ feet from the larger base?

Ans. 7 feet.

6. If the conical frustum in ex. 4 is placed with its smaller face on a rough horizontal plane which is gradually tilted up, the coefficient of friction being $\frac{1}{2}$, will the frustum slide or tumble?

Ans. It will slide. (See p. 185.)

7. A board of uniform thickness in the shape of a trapezium has its parallel sides 4 feet and 5 feet long and its breadth 9 inches; find the position of its centre of gravity.

Result. It is $4\frac{1}{3}$ inches from the longer side.

8. From a triangular piece of cardboard of uniform thickness is removed the inscribed circle; find the centre of gravity of the remainder.

Result. If I is the centre of the inscribed circle, and G the centre

of gravity of the complete triangle, the required point is on IG produced through G at a distance from G equal to

$$\frac{\pi \Delta}{s^2 - \pi \Delta} \cdot GI,$$

where Δ is the area and s is half the sum of the sides of the triangle.

9. At the vertices of a triangle are placed particles whose masses are proportional to the opposite sides; show that their centre of gravity is the centre of the inscribed circle.

Hence prove that the centre of gravity of three uniform rods of the same thickness and substance forming a triangle ABC is the centre of the circle inscribed in the triangle $A'B'C'$ (fig. 122) formed by the mid points of the sides.

10. A and B are two bodies of masses m and m'; A is fixed and B is made to rotate round a fixed axis; show that as B rotates the centre of gravity of the two together describes a circle.

[Let G and G' be the centres of gravity of A and B; then the centre of gravity, G'', of the two is on the line GG' dividing it in the ratio $m : m'$; but as G' describes a circle round the axis, its centre being O, suppose, G'' also describes a circle whose centre, P, lies on the line GO dividing it so that

$$OP : PG = m : m'.]$$

Hence if a box has a lid moveable round a hinge, and the lid is raised into various positions, the centre of gravity of the box and lid describes a circle.

11. A solid cube of side a is laid on the ground, and on its upper face is constructed a pyramid whose vertex is at a height h above this face on the vertical line through the centre of the cube; find the position of the centre of gravity of the whole solid thus formed. (The volume of a pyramid standing on any plane base is $\frac{1}{3}$ of the area of the base multiplied by the perpendicular height of the pyramid.)

Result. It is at a height of $\frac{1}{4}\,\dfrac{h^2 - 6a^2}{h + 3a}$ above the upper face.

12. On the base of a solid homogeneous hemisphere of radius r is erected a solid right cone of height h; find the position of the centre of gravity of the compound solid figure. (The centre of gravity of a solid hemisphere is $\frac{3}{8}r$ from the centre.)

Result. It is at a height of $\frac{1}{4}\,\dfrac{h^2 - 3r^2}{h + 2r}$ above the base.

13. Find the position of the centre of gravity of the surface of a cone including the surface of its base.

Result. If r is the radius of the base, h the perpendicular height, and l the slant height, the centre of gravity is at a distance above the base equal to

$$\frac{lh}{3(l+r)}.$$

14. ABC (fig. 128) is a board of uniform thickness in the shape of a right angled triangle, the right angle being C; it is placed with its plane vertical, the side AC resting on a rough horizontal plane; a cord is attached to B and pulled horizontally with a gradually increasing force; find whether the board will begin by sliding or toppling.

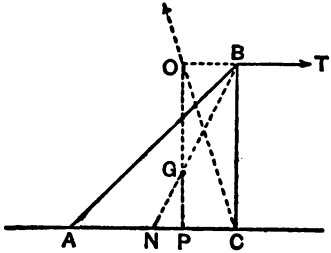


Fig. 128.

Let μ be the coefficient of friction, $b=AC$, $h=CB$. Then if the board is about to topple over C, it will be just in equilibrium under the action of *three* forces—viz. its weight W, the tension T, and the reaction of the ground. This last acts at C, because the board is out of contact with the ground everywhere except at C. Also these three forces must meet in a point. Hence by producing the cord to meet the vertical through the centre of gravity G, in O, the reaction must act in CO. But this requires the angle OCB to be $<\tan^{-1}\mu$; *i.e.* $\tan COP<\mu$.

Now $\tan COP=\frac{PC}{h}$, and since $CP:CN=BG:BN=2:3$ (where N is the mid point of AC), we have $CP=\frac{b}{3}$. Therefore for the possibility of toppling

$$\frac{b}{3h}<\mu.$$

If $\frac{b}{3h}>\mu$, the body must slide.

15. ABC (fig. 128) is a vertical section of a homogeneous triangular prism perpendicular to its edge at B and passing through its centre of gravity, the angle at C being right; the prism rests on the ground, for which $\mu=\frac{1}{3}$; its height BC is 6 feet, and $AC=8$ feet; a horizontal rope attached to B is pulled in the plane ABC with a gradually increasing force; find how equilibrium ceases and the corresponding tension in terms of W, the weight of the prism.

Result. The prism will slide (and not topple) when $T=\frac{1}{3}W$.

16. ABC (fig. 122, p. 194) is any triangle, A' being the middle point of BC; at A, C, and A' act three like parallel forces of 5, 3, and 9, respectively, and at B a force of 2 parallel to them but in the opposite sense; the forces being supposed to turn round these points, always remaining parallel, find the position of the centre of the system.

Result. If p and r are the perpendiculars from A and C on the opposite sides, the centre is distant $\frac{r}{2}$ from AB and $\frac{p}{3}$ from BC.

17. If in the last, in addition to the previously-given forces, a force of 5 acts at C' the mid point of AB in the sense opposite to the force 2 at B, and at B', the mid point of AC, a force 5 in the same sense as the force 2 at B, find the centre of the system.

Result. The centre is the centre of gravity of ABC.

18. $ABCD$ (fig. 94, p. 161) is a rectangle whose sides AB and AD are 12 and 5 inches long, respectively; at A, B, C, D act like parallel forces of 58, 27, 30, and 40 units, respectively; P is a point on DB such that $DP=4$ inches; R is a point on BD such that $BR=6$ inches; Q is a point on AC such that $AQ=3$ inches; at P, Q, R act forces parallel to the previous set, but in the opposite sense, of 39, 52, and 26 units, respectively; find the centre of the system, the forces being supposed to turn round these various points of application.

Result. O is their centre.

EXAMINATION ON CHAPTER XI

1. Define the *centre* of a system of parallel forces.
2. State the theorem of Plane-Moments for finding the distance of this point from any plane.
3. Define the *centre of gravity* of any body.
4. State the theorem of Plane-Moments for finding the distance of the centre of gravity of any number of bodies from any plane.
5. Given the centre of gravity of a body and that of any part of it, how do you find the centre of gravity of the remainder?
6. Where is the centre of gravity of a triangular plate of uniform thickness?
7. What system of equal particles has the same centre of gravity as a triangular area?
8. Given the distances of the vertices of a triangle from any plane, what is the distance of its centre of gravity from the plane?
9. Why must the bisectors of the sides of any triangle drawn from the opposite vertices meet in a point?
10. State the *particle method* of finding the position of the centre of gravity of the area of any polygon.
11. Give a simple construction for finding the position of the centre of gravity of the area of any quadrilateral. (It can be divided in two ways into two triangles.)
12. How do you find the centre of gravity of any solid homogeneous triangular pyramid? If the centre of gravity of each face is joined to the opposite vertex, why must the joining lines meet in a point?
13. What system of equal particles has the same centre of gravity as a triangular pyramid? What is the distance of the centre of gravity of a triangular pyramid from any plane in terms of the distances of the vertices from that plane?
14. Where is the centre of gravity of a solid homogeneous cone?
15. If any body is placed with a number of legs in contact with a horizontal plane, what is necessary in order that the body shall not topple over?

CHAPTER XII

MOTION IN A CIRCLE

52. Suppose a particle to move in a circle, $PQRS$ (fig. 129), being compelled to do so by any forces or any mechanism into the nature of which we need not inquire; then in each position, P, of the particle there is an acceleration along the normal, PC, to the path: there may or may not be an acceleration along the tangent.

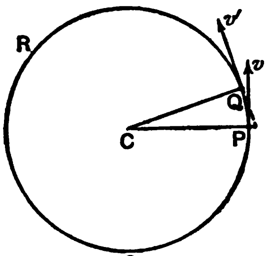


Fig. 129.

To find the acceleration along PC possessed by the particle when it is at P, take an extremely close position, Q, of the particle—the position which is reached by the particle at the end of $\frac{1}{1000}$ second, or any indefinitely small interval of time which we shall denote by t—and find the component of velocity along PC which the particle has at Q. Thus, let v = velocity at P, v' = velocity at Q; v' differs very little from v, and becomes equal to v if t is diminished to zero. Now if θ is the circular measure of the very small angle QCP,

$$\text{second component of vel. along } PC = v' \sin\theta = v' . \theta$$

$$\text{first} \quad ,, \quad ,, \quad = v \cos\frac{\pi}{2} = 0$$

$$\therefore \text{gain of vel. along } PC \quad = v' . \theta;$$

$$\therefore \text{time-rate of gain of vel. along } PC \quad = v' \frac{\theta}{t};$$

and if the radius PC is r, $\theta = \frac{PQ}{r}$; therefore

$$\text{time-rate of gain of vel. along } PC \quad = \frac{v'}{r} . \frac{PQ}{t}.$$

Now let t, and therefore PQ, diminish indefinitely; then $\frac{PQ}{t}$ is obviously v, and v' becomes v; so that

$$\text{time-rate of gain of vel. along } PC = \frac{v^2}{r}. \qquad . \quad . \quad (\alpha)$$

Time-rate of gain of velocity in any direction is acceleration in that direction; hence the acceleration along PC which the particle has at P is

$$\frac{v^2}{r};$$

and observe particularly that this acceleration is in the sense PC, that is, inwards towards the centre of the path—not outwards from the centre.

There may or may not be an acceleration at P along the tangent PT. If such exists, it is the time-rate of gain, $\frac{v' \cos \theta - v}{t}$, of velocity along PT when t is infinitely small—*i.e.* $\frac{v' - v}{t}$, since $\cos \theta = 1$, very nearly. Hence if the particle is moving round the circle with constant speed, there is no acceleration along the tangent at any point; but there is necessarily at every point an acceleration along the normal, since the particle has to move out of the tangent in order to describe the curved path.

Now by Newton's Second Axiom, wherever there is *acceleration* of a particle there must be *force*. Hence motion in a circle (or in any curved path) is impossible unless there is a component of force acting on the moving body *towards the concave side of the path*—*i.e.* towards its centre.

If N is the total component of all the forces acting on the particle in the sense PC when all these forces are resolved along the tangent PT and the normal PC, and if w is the weight of the particle, we have by (2), p. 22, the equation (N being taken in gravitation measure)

$$N = w\frac{v^2}{gr}. \qquad . \quad . \quad . \quad . \quad (\beta)$$

If N is taken in absolute measure, we simply omit g in the denominator, as already explained.

If there is a component of acceleration along PT denoted by a_s, and if S is the total component of force acting on the particle along PT, we have the equation

$$S = w\frac{a_s}{g}. \quad . \quad . \quad . \quad . \quad (\gamma)$$

The sense of a_s (and therefore of S) may be PT or TP; but the sense of N is absolutely fixed—viz. from P to C.

Suppose that a particle whose mass is $\frac{1}{2}$ ounce is moving in a circle of 6 inches radius with a velocity of $16^f/_s$ in a given position P; then the acceleration towards the centre, in this position, is $\frac{16^2}{\frac{1}{2}}{}^f/_{ss}$, the weight of the particle is $\frac{1}{32}$ pounds' weight, and therefore there must be acting on the particle an inward normal force of

$$\tfrac{1}{32} \cdot \frac{16^2}{32 \times \frac{1}{2}}, \text{ or } \tfrac{1}{2}, \text{ pounds' weight,}$$

in the position P.

We cannot say, from the above data, what force acts in the direction of the tangent PT, because we are not given anything about the way in which the velocity varies from point to point, so that we do not know the acceleration along PT.

EXAMPLES

1. A mass of 4 ounces is attached to one end of a flexible cord the other end of which is fixed on a smooth horizontal plane, the length of the cord being 3 feet; with what velocity must the mass be projected so as to cause a tension of 6 pounds' weight in the cord during the circular motion of the particle?

Ans. $v = 48\, {}^f/_s$.

2. In the last, with how many revolutions per minute must the mass be projected to cause a tension of 24 ounces' weight in the cord?

Ans. $\frac{240}{\pi}$.

3. There is a rough horizontal plate moveable round a vertical axis fixed through the face of the plate; a particle is placed on the plate at a distance of 18 inches from the axis, and the plate is then rotated with gradually increasing speed; find the greatest number of revolutions that it can make per second without causing the particle to slip, the coefficient of friction being $\frac{1}{3}$.

Result. $\frac{4}{3\pi}$ revolutions per second.

[Observe that since the only force acting on the particle in the plane of the plate is that of friction, if the particle moves with uniform velocity in a circle, the force of friction must act along the radius, and inwards towards the centre.]

4. An engine whose mass is 12 tons moves with a speed of 30 miles per hour on horizontal rails of the same height forming a curve of 600 feet radius; find the force exerted by the rails on the flanges.

Result. 1·21 tons' weight.

If we tie a stone to one end of a cord, holding the other end in the hand, and then whirl the stone round so that it describes a circle, the normal *inward force* which we know to be necessary for such motion is supplied by the cord: the stone is pulled inwards towards the hand by the tension, while the hand feels this force in the opposite (*i.e.* the *outward*) sense. Of course, as already pointed out in Chapter IV., there is no force in the universe which is not accompanied by its equal and opposite; but these equal and opposite forces do not act on *the same body*. The tension of the above cord acts on the *stone* as an inward force towards the concave side of the path of the stone; it is on the *hand* which holds the cord that it is felt as an outward force—a centre-flying force, or "centrifugal force," as it has long been commonly called.

This term "centrifugal force" has been the source of a very erroneous notion—the notion, namely, that whenever a particle is revolving in a circle it is acted upon by a force outwards, *from the centre*: that the particle has a tendency to fly outwards from the centre. It cannot be too strongly pointed out to the student that—

there is no such force acting on the particle, and there is no such tendency.

On the contrary, since the acceleration of a particle moving in a circle is by very geometrical necessity directed *inwards*, towards the centre, the resultant normal force on the particle is at each point of its path directed *inwards*, not outwards. Of course, on *some other body* (such as the hand, in the above case) this normal force is exerted in the outward sense, and it may, therefore, be properly called a centre-flying or "centrifugal" force. Such a justification of the use of the term "centrifugal force" is, however, contrary to the whole of our practice, in other respects, in Dynamics. For, when we are treating of the

equilibrium of a body under the action of its weight and pressures exerted upon it by walls or fixed planes, we do not pay any attention to the pull which *the Earth* experiences from the body and which is the exact equal and opposite of the *weight* of the given body. Why, then, introduce the term "centrifugal force" when we are considering the motion of a body, since we must admit that no such force is felt by the body? It is unnecessary to say that such a force is felt by *some* body; for this is no peculiarity of the force in question: every force in the universe has two opposite aspects, and we are certainly not in the habit of always paying attention to both of these aspects.

The practical objection to the use of the term "centrifugal force" when we are treating of the motion of a given body is that students are very apt to consider this force as one *which is acting on that body*; and this is a fatal error.*

As regards the tendency of a revolving body to fly outwards from the centre, no such thing exists. If in fig. 129, p. 207, CP is a cord fixed at C, and attached at P to a particle which is whirled round in the circle $PQRS$ on a smooth table, if the cord were to be suddenly cut in the position CP, the particle would simply move along the tangent PT with constant velocity, in accordance with Newton's First Axiom: it certainly would not move along CP outwards. In no other sense than this (a tendency to continue its motion at each point along the tangent) is there any tendency to recede from the centre.

A good example of motion in a curved path is furnished by a railway carriage going round a curved part of the line. At any instant a body moving in any curve may be considered for dynamical purposes as moving in a circle—viz. the circle which most nearly coincides with the curve at the position of the body, P. This is called the *circle of curvature* of the curve at the point P, and it can be practically drawn by drawing the normal at this point and the normal at a very close point, Q, on the curve; these two normals intersect in the centre, C, of the circle of curvature, and their common length, PC, is called the *radius of curvature* of the given curve at the point. If v is

* Thus, in a very suggestive and excellent book, "The Boy's Playbook of Science," by the late J. H. Pepper, *centrifugal force* is defined (p. 17) as "that force which drives a revolving body from a centre," and various illustrations are given—none of them, of course, proving any such thing.

the velocity of the moving body at P, and ρ is the length of the radius of curvature, the normal acceleration of the body at P is $\frac{v^2}{\rho}$; and if W is the weight of the body, there must be acting on the body a force in the direction and sense PC equal to

$$W\frac{v^2}{g\rho}, \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

in gravitation measure.

Now in the case of the railway carriage this normal force can be supplied as pressure against the flanges exerted by the rails. But it is undesirable to obtain the necessary normal force in this way because of the resulting wear of the flanges; and it is usual to obtain it by raising the end of the sleeper at which the outer rail is fixed, as represented in fig. 130. This is usually spoken of as "raising the outer above the inner rail."

The concave side of the path described by the carriage is at the left of the figure. Let the centre of curvature of the path of the centre of gravity, G, of the carriage be a point C on the horizontal line GH, and let $GC = \rho$. Let the outer rail, A, exert on the wheel a flange pressure X and a normal pressure Y in the plane of the wheel; then if X exists, there will be no flange pressure exerted by the inner rail, B, since the wheels do not fit *tightly* between the rails. Nevertheless we shall let X' and Y' be the corresponding forces exerted by the inner rail, observing that if X' exists, X is zero, and *vice versâ*. Let the carriage be moving with a speed v, and let i be the (very small) inclination of the sleeper to the horizon. Then, treating the whole carriage as a particle, its acceleration along the

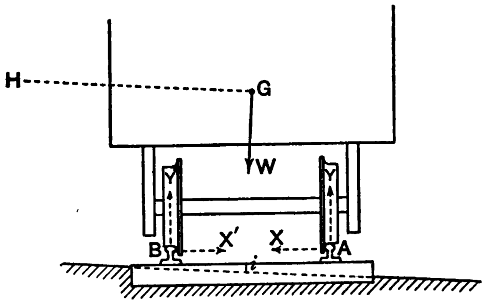


Fig. 130.

inward normal, GH, to the horizontal circle in which it is moving is $\frac{v^2}{\rho}$, so that there must be acting on it in the sense GH a component of force equal to $W\frac{v^2}{g\rho}$, by (β), p. 208, where W is the weight of the carriage.

Now the total component of force acting on the carriage along GH is $(X-X')\cos i+(Y+Y')\sin i$; hence

$$W\frac{v^2}{g\rho}=(X-X')\cos i+(Y+Y')\sin i. \quad . \quad (1)$$

Again, since the carriage has no vertical acceleration, there can be no vertical force; therefore

$$0=W+(X-X')\sin i-(Y+Y')\cos i. \quad . \quad (2)$$

These equations hold whatever the velocity may be. Now let v and i be so related that there is no flange pressure—*i.e.* let $X=0$ and $X'=0$. (Observe that, as above stated, *either* X or X' is always zero.) Then these equations give

$$\frac{v^2}{g\rho}=\tan i, \quad . \quad . \quad . \quad . \quad (3)$$

in which $\sin i$ or i (the circular measure) can be used instead of $\tan i$.

If the carriage is moving on this line with a speed, v', less than that given by (3), there will be flange pressure at the inner rail given by the equation

$$X'=W\frac{v^2-v'^2}{g\rho}, \quad . \quad . \quad . \quad (4)$$

and if it is moving with a speed, v', greater than that given by (3), there will be flange pressure, X, at the outer rail such that

$$X=W\frac{v'^2-v^2}{g\rho}. \quad . \quad . \quad . \quad (5)$$

EXAMPLES

1. If the radius of curvature of a railway line at a certain place is 400 yards, and trains travel there at the rate of 30 miles an hour, find the elevation of the outer above the inner rail so that there shall be no flange pressure, the gauge being $4\frac{1}{2}$ feet.

Result. About 2.7 inches.

2. If the gauge is a mètre and trains travel at 45 miles per hour at a part of the line where the radius of curvature is half a mile, find the elevation of the outer above the inner rail so that there shall be no flange pressure.

Result. About 5.16 centimètres.

3. If the radius of curvature of the line is 1089 yards, the gauge $4\frac{1}{2}$ feet, and the elevation of the outer rail 4 inches, what velocity will correspond to the absence of flange pressure?

Ans. 60 miles per hour.

4. If with the data in the last the speed of the train is 45 miles per hour and W its weight, what flange pressure is exerted?

Ans. A pressure by the inner rail equal to $\frac{7}{216}W$.

5. If the gauge is $4\frac{1}{2}$ feet, the elevation of the outer rail 3 inches, and the radius of curvature 363 yards, what speed corresponds to the absence of flange pressure?

Ans. 30 miles per hour.

6. If with the data in the last a train runs with a speed of 45 miles per hour, what flange pressure exists?

Ans. A pressure at the outer rail equal to $\frac{5}{72}W$, where W is the weight of the train.

Let us suppose the speed of a train going round a curve to increase continuously. Would the wheels leave the inner or the outer rail? As already shown in equation (5), increase of speed above that which corresponds to the absence of flange pressure, produces pressure on the *outer* rail; hence the wheels cannot leave the outer rail. It is otherwise evident from the direct application of Newton's Second Axiom as expressed in (β), p. 208, that the outer wheels could never leave the rail, because if they did, we could not obtain inward normal component of force which is always necessary to make a particle describe a curved path.

The velocity which would just produce the lifting of the inner wheels off the rail can be easily found. For we have now $X'=0$, $Y'=0$ in (1) and (2); therefore

$$W\frac{v^2}{gp} = X\cos i + Y\sin i, \quad . \quad . \quad . \quad (6)$$

$$0 = W + X\sin i - Y\cos i. \quad . \quad . \quad (7)$$

To these must be joined the equation which expresses the fact that the carriage has no appreciable angular motion about G; that is, we can assume that the forces acting have no moment about a horizontal axis through G parallel to the

direction of motion. If h is the height of G above the plane AB, the gauge AB being $2a$, we have, then,

$$hX = aY, \quad . \quad . \quad . \quad . \quad (8)$$

and from these equations we have

$$\tan i = \frac{hv^2 - agp}{av^2 + hgp}. \quad . \quad . \quad . \quad . \quad (9)$$

This relation holds, then, when the inner wheel is about to rise off the rail: it is usually described as the condition that the carriage should "upset"; but obviously it has nothing to do with "upsetting." Upsetting over the outer rail would mean something very different.

Conical Pendulum.—Let one end, O, fig. 131, of a cord be fixed, and to the other end, P, let a particle of weight W be attached; let the cord be drawn out from the vertical OC and the particle projected horizontally at right angles to the plane COP with such a velocity that it goes round in a horizontal circle PBA, the cord OP describing a right cone round the line OC; it is required to find the requisite velocity, v, of projection.

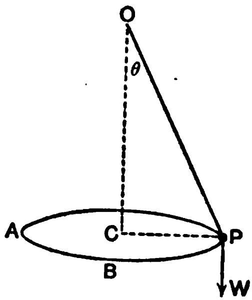


Fig. 131.

Since the particle moves in a horizontal plane, it has no vertical motion, therefore there is no vertical force, and we have, if T is the tension of the cord,

$$T\cos\theta - W = 0. \quad . \quad (10)$$

Let $l = OP$; then P describes a circle of radius $l\sin\theta$ with velocity v, therefore its acceleration towards C, the centre of this circle, is $\frac{v^2}{l\sin\theta}$. Also the component of force along PC, towards C, acting on the particle is $T\sin\theta$, or $W\tan\theta$; hence by (β), p. 208,

$$W\frac{v^2}{gl\sin\theta} = W\tan\theta,$$

$$\therefore \; v^2 = gl\sin\theta\tan\theta, \quad . \quad . \quad . \quad (11)$$

which gives the velocity necessary for the circular motion. The time of a complete revolution is $2\pi\sqrt{\dfrac{l\cos\theta}{g}}$.

A cord and particle moving in this way constitute a *conical pendulum*.

EXAMPLES

1. If $OP=26$ inches and P is drawn out 10 inches from the vertical OC, find the velocity of projection necessary to make P describe a horizontal circle.

Result. $3\frac{1}{3}$ feet per second.

2. A conical pendulum whose length is 7 inches makes 2 revolutions per second, what is its angular deviation from the vertical? (Take $\pi=\frac{22}{7}$.)

Ans. $\cos^{-1}\frac{42}{121}$.

3. A mass of 34 ounces on a rough horizontal plate for which the coefficient of friction is $\frac{1}{2}$ is attached to a flexible cord which passes through a hole in the plate and has a mass of 8 ounces attached to its other extremity, this mass hanging freely at a distance of 4 feet below the hole; what is the greatest velocity that can be given to the second body so that it can move as a conical pendulum?

Ans. $\dfrac{60}{\sqrt{17}}$ f/s.

4. Generally, if P is the mass on the plate, μ the coefficient of friction, Q the hanging mass, and l the length of the cord at the pendulum, what is the greatest velocity at Q?

Ans. $\left(\dfrac{\mu^2P^2-Q^2}{\mu PQ}gl\right)^{\frac{1}{2}}$.

5. A hollow smooth cone of semi-vertical angle α is placed with its axis vertical and vertex downwards; if a marble is projected horizontally along the inner surface with a velocity v, at what distance must it be from the vertex so that it shall describe a circle?

Ans. At a vertical height $\dfrac{v^2}{g}$ above the vertex.

6. If in example 4 the plate is smooth and P is placed at a distance a from the hole, with what velocity must P be projected so that it shall describe a circle, Q hanging vertically at rest?

Ans. $\left(\dfrac{Q}{P}ga\right)^{\frac{1}{2}}$.

7. If in the last example P is projected with any velocity, u, so as to describe a circle while Q moves as a conical pendulum, what must Q's velocity be?

Ans. $\left(\dfrac{Pl}{Qa}u^2-\dfrac{Q g^2 a l}{P u^2}\right)^{\frac{1}{2}}$.

Revolving Ring.—Suppose that a ring, hoop, or band, AB, (fig. 132), of any material revolves rapidly in its plane about an axis through its centre, C, perpendicular to this plane, then at each point of its circumference there is produced a tension which tends to tear the body asunder.

Imagine a small length, PQ, of the ring contained between two very close cross-sections made by planes through C perpendicular to the plane of the ring, and consider the *separate motion* (see p. 51) of this element. The element describes a circle with constant speed, v.

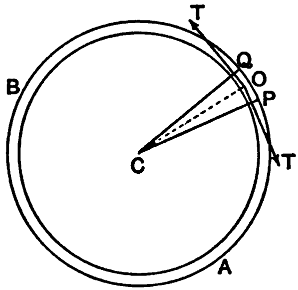


Fig. 132.

Now what are the forces exerted on this element by the portions of the material in contact with it? Over the whole of the cross-section at P is exerted a pull or tension along the tangent —*i.e.* at right angles to CP; let the whole amount be T. At Q there is an equal force perpendicular to CQ. Let the lines of action of these two forces meet in O, and take the sum of their components in the sense OC. If θ is the circular measure of the very small angle PCQ, this sum is $T\sin PCO + T\sin QCO$, that is

$$T.\theta \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now if W is the weight of the whole ring and r its mean radius, its weight per unit length is $\frac{W}{2\pi r}$, therefore the weight of the element PQ is $\frac{W}{2\pi r} \times PQ$, or $\frac{W}{2\pi}.\theta$. Hence by (β), p. 208,

$$\frac{W}{2\pi}.\theta.\frac{v^2}{gr} = T.\theta$$

$$\therefore T = W.\frac{v^2}{gl}, \quad . \quad . \quad . \quad (13)$$

where l = the mean length of the ring.

The breaking of the ring depends on the *intensity* of the stress exerted on the cross-section — *i.e.* on the amount of

tension per unit area of cross-section. If s is the area of the cross-section, say in square inches, and T is measured in tons' weight, $\frac{T}{s}$ is the intensity of the stress in tons' weight per square inch, so that

$$\frac{T}{s}=\frac{W}{s}\frac{v^2}{gl}. \qquad . \quad . \quad . \quad . \quad (14)$$

We must be very careful about units in (13). If g is taken in feet per second per second, v must be taken in feet per second, and l in feet; T will then be in the same units as W.

EXAMPLES

1. A ring of metal whose mass per cubic foot is 480 pounds has a mean radius of 1 foot and a cross-section of 2 square inches; it revolves round an axis through its centre perpendicular to its plane, making 10 revolutions per second; find the intensity of tangential stress produced in the ring.

The velocity of a point on the ring is $2\pi \times 10$ f/s, $\therefore v=20\pi$;

$$W=2\pi \times \frac{2}{144} \times 480 \text{ pounds}=\frac{40\pi}{3}; \; l=2\pi \text{ feet, therefore}$$

$$T=\frac{40\pi}{3} \cdot \frac{400\pi^2}{32 \times 2\pi} \text{ pounds' weight}$$

$$=\frac{250\pi^2}{3}=822.4 \qquad ,, \qquad ;$$

therefore the intensity of the stress is

$$\frac{822.4}{2}=411.2 \text{ pounds' weight per square inch.}$$

2. Supposing that the greatest intensity of stress that the material of the above ring will stand is 5 tons' weight per square inch, what is the greatest number of revolutions per second that can be given to it?

Assume $\frac{T}{s}$ in (14) to be equal to the limiting intensity, viz. 5×2240 pounds' weight per square inch; then

$$5 \times 2240=\frac{40\pi}{3 \times 2} \cdot \frac{v^2}{32 \times 2\pi}$$

$$\therefore v=32\sqrt{105}=320.6 \text{ f/s.}$$

If the ring makes n revolutions per second, $v=2\pi n$ f/s; therefore

$$n=\frac{320.6}{2\pi}=51, \text{ approximately.}$$

3. A circular hoop 3 feet in diameter and 2 inches broad is rotating round its axis at the rate of 10 revolutions per second; find, in tons' weight per square inch, the intensity of tangential stress, the material having a mass of 480 pounds per cubic foot.

Result. About .413.

Simple Harmonic Motion.—If a point P (fig. 133), moves along a circle with constant speed, V, the foot, Q, of the perpendicular from P on any fixed diameter, AB, of the circle moves with variable speed along this diameter, and the motion of Q is called *simple harmonic motion.*

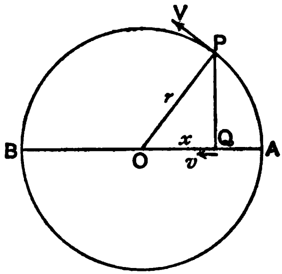


Fig. 133.

Let V be the speed of P; then the speed, v, of Q is $V \sin POQ$, if O is the centre. If $OQ = x$ and r is the radius of the circle, we have

$$v = \frac{V}{r} \cdot \sqrt{r^2 - x^2}. \quad . \quad (15)$$

Now when P has moved along the circle from A to B, Q has moved over AOB; but the time taken by P in this motion is $\frac{\pi r}{V}$, or $\frac{\pi}{\frac{V}{r}}$. Hence this is the time taken by Q to go from A to B. When P leaves B, Q moves back towards A. The motion of Q from A to B and back to A is called a *complete oscillation*: it is double the above value. The motion from A to B is a semi-oscillation.

The velocity of Q is a maximum at O, and is zero at A and at B.

Hence if a point moves along a right line so that its speed is expressed by the equation

$$v = k\sqrt{a^2 - x^2}, \quad . \quad . \quad . \quad . \quad (16)$$

where k is a constant and x is at any instant the distance of the moving point from its position, O, of greatest velocity, its velocity will be zero at each of the two points, A and B, distant a from O at the right and left. The motion will

oscillate between these two points, and will be simple harmonic. Moreover, the time of a semi-oscillation is

$$\frac{\pi}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

as we see by a comparison with (15). In fact, if on AB as diameter we describe a circle and cause a point, P, to travel round this circle with a speed equal to ka, we have two points, P, Q, related exactly as in fig. 133.

If a point moves on *any* curve in such a way that its velocity is expressed by the equation

$$v = k\sqrt{a^2 - s^2}, \quad . \quad . \quad . \quad . \qquad (18)$$

where s is at any time the arc distance of the moving point, P, from some fixed point, O—*i.e.* the length of the arc OP of the curve—and a and k are constants, the motion is simple harmonic, and the time of a semi-oscillation is (17).

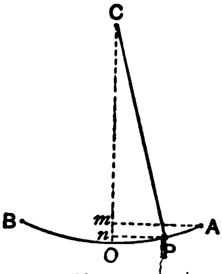


Fig. 134.

Simple Pendulum.—If a small particle, P (fig. 134), is attached to one end of a very fine flexible cord the other end, C, of which is fixed, and the particle swings freely in a vertical plane, we have what is called a *simple pendulum.* Let the cord be drawn out from the vertical into the position CA and then let go; draw Am and Pn perpendicular to CO, the vertical line through C. Then if v is the velocity of the particle at P, we have, as shown at p. 106,

$$v = \sqrt{2g \cdot mn}. \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Let l be the length of the cord (in feet if g is taken in $^{f}/_{ss}$; in centimètres if g is in $^{c}/_{ss}$); then by Prop. VIII. of Euclid B. VI., we have $2l \times Om = OA^2$, where OA means the chord of the arc OA. Also $2l \times Pn = OP^2$, where OP means the chord of the arc OP; hence

$$mn = \frac{OA^2 - OP^2}{2l}. \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

This is true no matter how large the arcs OA and OP are. Now let these arcs be very small compared with l, that is,

let the angle OCA be very small — say an angle of 5° or even 10°. In this case the length of the chord and the length of the arc are, to a high degree of accuracy, equal; so that, if the arcs OA and OP are denoted by a and s, respectively, (20) gives

$$mn = \frac{a^2 - s^2}{2l},$$

and (19) gives

$$v = \sqrt{\frac{g}{l}} \cdot \sqrt{a^2 - s^2}. \qquad . \quad . \quad . \quad (21)$$

Now comparing this with (18), we see that the motion of P is simple harmonic, the time of a semi-oscillation being given by the equation

$$t = \pi \sqrt{\frac{l}{g}}. \qquad . \quad . \quad . \quad . \quad . \quad (22)$$

Taking $OB = OA$, the points of zero velocity are A and B, and (22) expresses the time of motion from A to B. A *complete* oscillation is the time from A to B and back to A: it is double the value (22).

A *seconds' pendulum* is such that the time from A to B is one second. Its length is obtained by putting $t = 1$ in (22); that is, the length of a seconds' pendulum is

$$\frac{g}{\pi^2},$$

which is about 39.1 inches at a place where $g = 32.2\,{}^{f}/_{ss}$.

The tension, T, of the cord in any position is easily found by (β), p. 208. For, if v is the velocity of P and the weight of the particle is w, the total normal component of force acting on the particle in the inward sense, PC, is $T - w\cos\theta$, where $\theta =$ angle OCP; hence

$$T - w\cos\theta = w\frac{v^2}{gl}. \qquad . \quad . \quad . \quad (23)$$

But $v^2 = 2g\,.\,mn = 2g(Cn - Cm) = 2gl(\cos\theta - \cos\alpha)$, where $\alpha =$ angle OCA. Substituting this value of v in (23), we have

$$T = w(3\cos\theta - 2\cos\alpha). \qquad . \quad . \quad . \quad (24)$$

If the particle were projected along the circle at A with a velocity u, we should have

$$v^2 = u^2 + 2g \, . \, mn,$$

and (23) would become

$$T = w\left(\frac{u^2}{gl} + 3\cos\theta - 2\cos\alpha\right). \quad . \quad . \quad (25)$$

This holds, of course, whatever be the magnitudes of θ and α—not merely for the small displacement of an ordinary pendulum.

EXAMPLES

1. If the length of a seconds' pendulum is increased by $\frac{1}{100}$ of its value, how many beats will it lose in 24 hours?

If the length, l, of any simple pendulum is altered to $l(1+\frac{1}{100})$, or $\frac{101}{100}l$, and if t' is the new time of oscillation,

$$t' = \pi\sqrt{\frac{l}{g}} \cdot \frac{\sqrt{101}}{10}.$$

In 24 hours there are 86400 seconds, therefore the number of beats made in this time is $\frac{86400}{t'}$, or $\frac{86400}{\pi}\sqrt{\frac{g}{l}} \cdot \frac{10}{\sqrt{101}}$. Before the alteration of length the number was $\frac{86400}{\pi}\sqrt{\frac{g}{l}}$; hence the loss is

$$\frac{86400}{\pi}\sqrt{\frac{g}{l}}\left(1 - \frac{10}{\sqrt{101}}\right) \text{ beats.}$$

For a seconds' pendulum $\pi\sqrt{\frac{l}{g}} = 1$; therefore the loss of beats is 429.

2. At a place where $g = 32.2 \; {}^{f}/_{ss}$ a seconds' pendulum is lengthened by 1 inch, what is the new time of oscillation?

Ans. 1.0127 seconds.

3. If a pendulum which oscillates in 1 second at the Equator gains 300 beats in 24 hours when taken to the Pole, compare the value of g at the Pole and the Equator.

Result. $\frac{g \text{ at Pole}}{g \text{ at Equator}} = \left(\frac{289}{288}\right)^2.$

4. If a pendulum beating seconds at the base of a mountain 5 miles high is taken to the top, how many beats will it lose in 24 hours, supposing the the radius of the Earth to be 4000 miles and that g varies inversely as the square of the distance from the centre?

Ans. 108.

5. A particle suspended vertically by a cord of length l is projected horizontally from the lowest point, what is the least velocity of projection that will cause the particle to go completely round in a circle?

Ans. $\sqrt{5gl}$. [Observe that in (25), $\alpha=0$, and $\therefore$ $T=w\left(\frac{u^2}{gl}-2+3\cos\theta\right)$. Now T is never to become negative; but the greatest danger of a negative value occurs when $\theta=\pi$; then we must have $\frac{u^2}{gl}-5$ positive; therefore, etc.]

6. A particle whose mass is 50 grammes is attached to one end of a cord 60 centimètres long, the other end of which is fixed; the cord is kept tight in a horizontal position and the particle is projected downwards with a velocity of 20 mètres per second; find the tension of the cord in the lowest position of the particle.

Result. 3548 grammes' weight.

7. In the last what is the least velocity that would cause the particle to go completely round in a circle?

Ans. 420.21 $^c/_s$.

EXAMINATION ON CHAPTER XII

1. If a point moves in a circle of radius r with velocity v, what is its acceleration at each point toward its centre?
2. If a point moves in a curved path must it have an acceleration along the tangent at each point? Must it have an acceleration along the normal?
3. If *all* the forces acting on a particle, when resolved into components along and perpendicular to the tangent to a curve at the position of the particle, give a normal component towards the *convex* side of the curve, can the particle move in this curve under the action of these forces alone?
4. What fallacy is sometimes (if not usually) implied in the notion of "centrifugal force"?
5. If a particle moves in any curve, has it a "tendency to fly outwards"?
6. If a particle is attached to one end of a cord and whirled round in a horizontal plane, and the cord suddenly breaks, what does the particle do?
7. In the case of a railway carriage going round a curved part of the line, what supplies the necessary component of force towards the centre of curvature of the line?
8. If the speed of the carriage were continuously increased, could the *outer* wheel lift off the rail?

9. What plan is adopted to obtain the necessary normal inward component of force on a train going round a curve without calling flange pressure into play?

10. When a metal ring revolves round an axis through its centre perpendicular to its plane why is there stress produced at each point in it?

11. Define *simple harmonic* motion.

12. What is the time of a small oscillation of a simple pendulum?

13. How can the values of g at different places be compared by means of a simple pendulum?

CHAPTER XIII

MOTION OF PROJECTILES

53. If a particle is projected in any direction while its motion is subject to a constant acceleration in some other direction, the path of the particle is a parabola.

Suppose that a particle is projected along the line OA (fig. 135) with any velocity u f/s, and that the particle is acted upon by a constant force in the fixed direction OB throughout its motion; then we know that its motion has a constant acceleration in the direction OB, since constant force produces constant acceleration in a particle (p. 20). Let this acceleration be a f/ss. Then, if at the end of any time, t, the particle is found at the point P, we may consider that the particle has moved over a distance, ON, in the direction of OA, which is due to the velocity u, and at the same time has gone over a distance, OM, in the direction OB, which is due to the acceleration a; and $OMPN$ is a parallelogram in which

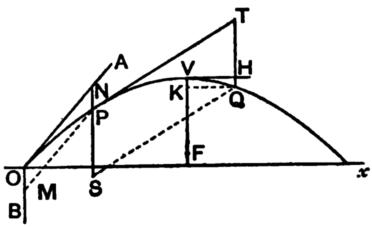


Fig. 135.

$$ON = ut, \quad . \quad . \quad . \quad . \quad (1)$$
$$\text{and } OM = NP = \tfrac{1}{2}at^2. \quad . \quad . \quad . \quad . \quad (2)$$

Hence, no matter what the value of t may be, the position, P, of the particle is defined by the two oblique distances PN and PM from the fixed lines OA and OB, these distances being connected by the equation

$$PM^2 = \frac{2u^2}{a} \cdot PN \quad . \quad . \quad (3)$$

The curve, OPQ, which is defined by this equation is a parabola whose axis is parallel to OB, the line OA being a tangent at O.

At the end of any time, t, the particle has a velocity at parallel to OB and a velocity u parallel to OA. Of these two is the resultant velocity compounded. This resultant velocity takes place along the tangent PT to the parabola in each position of the particle. Of course this velocity can be used instead of u in equation (3), the lines of reference OA and OB being replaced by the tangent PT and the line PS parallel to OB. That is, if u' is the velocity at P along PT, and Q any other position of the particle, we have

$$QS^2 = \frac{2u'^2}{a} . QT, \qquad . \quad . \quad . \quad (4)$$

$PSQT$ being a parallelogram, exactly analogous to the parallelogram $OMPN$. This requires no special proof, because we can, of course, consider the particle at P as projected along the line PT with velocity u', and having the acceleration a in the direction PS—*i.e.* we can regard P as the origin of projection instead of O.

Equation (3) or (4) may be regarded as the equation of the parabola referred to the oblique axes formed by the tangent at *any* point and the diameter at this point.

The point, V, at which the tangent, VH, is at right angles to the diameter, VK, is the vertex of the parabola; and if QH and QK are the perpendiculars on these lines from any point, Q, on the curve, and U is the resultant velocity at V (*i.e.* the velocity along VH), we have

$$QK^2 = \frac{2U^2}{a} . QH. \qquad . \quad . \quad . \quad (5)$$

Comparing this with the equation

$$y^2 = 4ax, \qquad . \quad . \quad . \quad . \quad (6)$$

which is the equation of a parabola referred to the diameter and tangent at its vertex, we see that the distance, a, of the focus, F, from V is

$$\frac{U^2}{2a} \qquad . \quad . \quad . \quad . \quad . \quad (7)$$

The velocity, U, at V is easily found in terms of the velocity, u, at O and the direction of OA. For, let OA make the angle ϵ with a perpendicular, Ox, to OB; then since at every point we may consider the resultant velocity as compounded of the velocity u in the direction OA and the velocity at in the direction OB, if the resultant of these two oblique components is perpendicular to OB (as it is at V), we have for this resultant

$$U = u\cos\epsilon.$$

Also for the time, t, of reaching the vertex V, after the projection at O, we have

$$at = u\sin\epsilon \quad \therefore \quad t = \frac{u\sin\epsilon}{a}. \qquad (8)$$

Hence also the distance of the focus from V is

$$\frac{u^2\cos^2\epsilon}{2a} \qquad (9)$$

This would be exactly the state of affairs if a particle were projected in any direction and acted upon by no other force than its *weight*. *Actually* such a particle would throughout its motion be subject to another force which is variable in both magnitude and direction—viz. the resistance of the air—and the calculation of its path, if we had to take account of this variable force, would be a matter of extreme mathematical difficulty. We do not know even the law according to which the air resistance varies: it has lately been supposed, as a result of experiments, that, so far as the resistance depends on the velocity of the particle, it may be taken to vary as the *cube* of the velocity. But if the velocity of projection is not very great, and the particle, or *projectile*, is one of considerable weight, the resistance of the air can be neglected. It is found that for velocities of projection not exceeding $500\,{}^f/_s$, if the projectile is a massive one, the resistance of the air has not much influence on the motion.

We shall assume that this resistance is negligible, and that a projectile is acted upon continuously by its weight alone. The result is that the vertical motion is uniformly accelerated in the downward vertical direction, and that the horizontal motion is not accelerated.

What has just preceded holds here, the acceleration a being g in the vertical direction OB. The path is a parabola; the vertex of the path is reached in the time

$$\frac{u \sin \epsilon}{g} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

after projection at O, ϵ being the *angle of elevation*, as it is called—*i.e.* the angle made with the horizon by the direction, OA, of the velocity of projection; and the distance of the focus from the vertex is

$$\frac{u^2 \cos^2 \epsilon}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We may present the calculation in a slightly different way, by considering separately the horizontal and the vertical motions of the projectile. Thus, at the point, O, of projection (fig. 136) draw the horizontal and vertical lines Ox and Oy; let u be the velocity of projection at O and ϵ the angle of elevation. Resolve u into the horizontal component u_x, or $u \cos \epsilon$, and the vertical component u_y, or $u \sin \epsilon$. Then if P is the position of the particle after the time t, and v_x the horizontal component and v_y the vertical upward component of velocity at P, we have

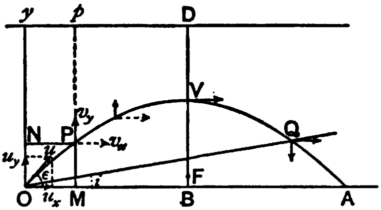


Fig. 136.

$$\left.\begin{aligned} v_x &= u \cos \epsilon, \quad . \quad . \quad . \quad . \quad . \quad (12) \\ v_y &= u \sin \epsilon - gt, \quad . \quad . \quad . \quad . \quad (13) \end{aligned}\right\}$$

because since there is no horizontal force acting on the particle, there is no horizontal acceleration, and therefore the horizontal component of velocity is the same at all points, O, P, V, Q, A of the path; and since the only acceleration is g downwards, the vertical component must have the value given by equation (13). [See also p. 77.]

Again, if PM is the perpendicular from P on Ox, and $PM=y$, $OM=x$ we have, since the horizontal and vertical motions can be considered as quite separate,

$$\left.\begin{aligned} x &= u \cos \epsilon \,.\, t \quad . \quad . \quad . \quad . \quad . \quad (14) \\ y &= u \sin \epsilon \,.\, t - \tfrac{1}{2} g t^2. \quad . \quad . \quad . \quad (15) \end{aligned}\right\}$$

In fact, if PN is drawn perpendicular to Oy, and if we imagine one particle projected at O along the horizontal plane OA with the velocity $u \cos \epsilon$, and at the same instant another particle projected along Oy with the velocity $u \sin \epsilon$, after any time, t, the particle which was projected from O with velocity u and elevation ϵ will have the x of the first and the y of the second particle; *i.e.* when the first is at M, the second is at N, and from these two positions the position, P, of the particle in the parabola is compounded.

As represented in the figure, the horizontal component of velocity remains equal to u_x in all positions, and the vertical component alone alters, gradually diminishing until the vertex, V, is reached, and then assuming a downward sense.

The principle of work and energy shows at once that when the particle again reaches the horizontal plane drawn through O—as it does at A—the velocity becomes equal to u, because the weight has done a zero total of work from O to A on the particle, therefore the velocity at A is equal to that of O.

To find the distance OA, or the *range* on the horizontal plane through O, put $y=0$ in (15). This gives

$$(u \sin \epsilon - \tfrac{1}{2}gt)t = 0,$$

which gives $t=0$, or

$$t = \frac{2u \sin \epsilon}{g}. \quad . \quad . \quad . \quad . \quad (16)$$

The value $t=0$ corresponds to the zero value of y at O; the value (16) is that which corresponds to A, and is called the *time of flight.*

The distance OA is obtained by substituting this value of t in (14), so that

$$OA = \frac{2u^2 \sin \epsilon \cos \epsilon}{g} = \frac{u^2}{g} \sin 2\epsilon \quad . \quad . \quad (17)$$

$$= \frac{2u_x \cdot u_y}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now (17) shows that, with a given velocity, u, of projection the range will be greatest when the elevation is 45°; and, moreover, that to produce a *given* range OA there are *two* directions of projection, these being equally inclined to the direction for greatest range; for, let one direction make ϵ

with Ox, and another make ϵ with Oy; then if the velocity of projection is u for both, the range for the second is, by (17)

$$\frac{u^2}{g}\sin 2\left(\frac{\pi}{2}-\epsilon\right), \text{ which} = \frac{u^2}{g}\sin 2\epsilon,$$

and is therefore the same as for the first.

To find the time of reaching V, and the height of V above OA, observe that at V we have $v_y = 0$, $\therefore$ by (13), $t = \frac{u \sin \epsilon}{g}$, which might have been inferred from (16) since the time of reaching V is half that of reaching A. Substituting $t = \frac{u \sin \epsilon}{g}$ in (15), we have for the greatest height attained

$$VB = \frac{u^2 \sin^2 \epsilon}{2g}. \quad . \quad . \quad . \quad (19)$$

The equation of the parabola referred to Ox and Oy as axes is obtained by substituting in (15) the value of t obtained from 14; thus we have

$$y = x \tan \epsilon - \frac{gx^2}{2u^2}\sec^2 \epsilon. \quad . \quad . \quad . \quad (20)$$

We have already seen in (9) that the distance of the focus, F, from V is $\frac{u^2}{2g}\cos^2 \epsilon$, and therefore the equation of the curve referred to horizontal and vertical axes at V (the axis of y being VB) is

$$x^2 = \frac{2u^2 \cos^2 \epsilon}{g} \cdot y. \quad . \quad . \quad . \quad (21)$$

These values of the time of flight, greatest height attained, etc., are at once obvious if we consider the position P of the particle at any time as compounded of the positions, M and N, of two particles, one moving along Ox in consequence of an initial velocity u_x, and the other moving along Oy in consequence of an initial velocity u_y combined with the downward acceleration g. Thus, the upward velocity of the second will be destroyed in the time $\frac{u_y}{g}$, and when this happens the projectile is at V; therefore the time of

flight is obviously $2\dfrac{u_y}{g}$. Again this particle, N, will attain the height $\dfrac{u_y^2}{g}$ when its upward velocity ceases; and the equation of the parabola referred to the tangent and diameter at V is by (5)

$$x^2 = \frac{2u_x^{\,2}}{g}.y.$$

As a numerical example, suppose that the particle is projected from O in a direction making $\tan^{-1}\frac{5}{12}$ with OA with a velocity of $390\,{}^f/_s$. Then $u_x = 360\,{}^f/_s$, $u_y = 150\,{}^f/_s$; the greatest height attained $= \dfrac{u_y^2}{2g} = 351\frac{9}{16}$ feet; time of flight $= 2\dfrac{u_y}{g} = 9\frac{3}{8}$ seconds.

In what direction and with what velocity will this particle be moving after 5 seconds?

Consider the values of v_x and v_y at the time $t = 5$. We have

$$\begin{aligned} v_x &= u_x = 360, \\ v_y &= u_y - gt = -10. \end{aligned}$$

Hence the particle has exhausted its upward velocity and is moving downwards in its path, having passed the highest point, V; its vertical downward velocity $= 10\,{}^f/_s$; and the tangent of the angle which the direction of its motion (tangent to the path) makes with the horizon is $\dfrac{v_y}{v_x} = \frac{1}{36}$; its resultant velocity $= \sqrt{v_x^2 + v_y^2} = 10\sqrt{36^2 + 1} = 10\sqrt{1297}\,{}^f/_s$. The focus is below the horizontal plane Ox at a distance $\dfrac{u_x^2 - u_y^2}{2g} = 1673.44$ feet.

Sometimes it is desired to find when, where, and with what velocity the projectile will strike an inclined plane passing through the point of projection.

Thus, with the above numerical values, find the point in which the projectile will strike the inclined plane OQ whose inclination to the horizontal plane Ox is $\tan^{-1}\frac{1}{4}$. Let it strike

the plane at Q; then if QS is the perpendicular from Q on Ox, we have $QS = \frac{1}{4} . OS$, and we have merely to put

$$y = \tfrac{1}{4}x$$

in (14) and (15). Therefore

$$150t - 16t^2 = \tfrac{1}{4} \times 360t,$$

which gives $t = 0$ or $t = \frac{15}{4}$. The value $t = 0$ corresponds to the position O on the plane OQ; the value $t = \frac{15}{4}$ corresponds to Q. Hence, putting $t = \frac{15}{4}$ in (12), (13), (14), (15), we have

$$v_x = 360; \quad v_y = 30,$$
$$x = 1350; \quad y = 337\tfrac{1}{2}.$$

Hence the particle is still on the upward slope of its path, the tangent to the path at its position making $\tan^{-1}\frac{30}{360}$, or $\tan^{-1}\frac{1}{12}$, with the horizon, so that the angle at which it strikes OQ is $\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{12}$, or $\tan^{-1}\frac{48}{149}$; the distance $OQ = 1350 \times \sec QOx$, or $1350 \times \frac{13}{12}$, or $1462\frac{1}{2}$ feet.

EXAMPLES

1. If a particle is projected horizontally from the top of a cliff, prove that it will take the same time to strike the sea whatever the velocity of projection may be.

Let O (fig. 137) be the point of projection, Ox and Oy the horizontal and vertical lines at O, the latter drawn downwards; let OC be the height of the cliff above the sea, whose surface is CBE.

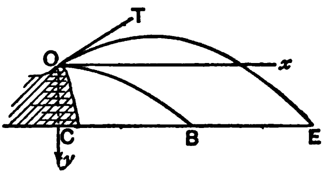


Fig. 137.

Then since we can consider the horizontal and vertical motions separately, if the particle is projected in the direction Ox, it has no vertical velocity, and it has therefore to move through the vertical distance OC with no velocity at starting but with the downward acceleration g. Hence for the time, t, of moving through the height OC, we have

$$OC = \tfrac{1}{2}gt^2, \quad \therefore t = \sqrt{\frac{2OC}{g}}.$$

If, then, OB is the path described by the particle, the time taken to reach B along this path is the same as would be taken to reach C if the particle were dropped at O and could move down OC.

2. If the particle is projected horizontally from O with a velocity of $500\ {}^f/_s$ and strikes the sea after 4 seconds, what is the height of the cliff, and what is the distance OB?

If h is the height OC, we have $h = \frac{1}{2}gt^2 = 16 \times 16 = 256$ feet.

The horizontal motion takes place with the constant velocity $500\ {}^f/_s$, therefore $CB = 500 \times 4 = 2000$ feet; $\therefore\ OB = \sqrt{256^2 + 2000^2}$ $= 2016.3$ feet.

3. If the particle is projected from O in a direction, OT, making with the horizon $\tan^{-1}\frac{5}{12}$ with a velocity of $390\ {}^f/_s$, and it strikes the sea 10 seconds after projection, find the height of the cliff and the horizontal distance CE.

Taking the positive direction of the axis of y vertically downwards, and resolving the velocity $390\ {}^f/_s$ into horizontal and vertical components, we have the horizontal component equal to 360 and the vertical (upward) component 150. Hence if x, y are the co-ordinates of the particle at any time t, we have

$$x = 360t$$
$$y = 16t^2 - 150t.$$

Putting $t = 10$, y will be OC, and this is equal to 100, while $x = CE = 3600$.

4. If the particle is projected in the direction OT which makes $\tan^{-1}\frac{8}{15}$ with the horizon with a velocity of $340\ {}^f/_s$, and the height OC of the cliff is 225 feet, find when and where it will strike the sea.

Taking axes as in the last, the horizontal and vertical velocities of projection are 300 and -160. Hence

$$x = 300t,$$
$$y = 16t^2 - 160t.$$

Now putting $y = 225$, we find $t = \frac{45}{4}$ or $-\frac{5}{4}$. The second of these values corresponds to the point in which the path EO produced backwards through O cuts the line CE; the first is the value which we require and it is the time of reaching E. Substituting it in the value of x, we have

$$x = CE = 3375 \text{ feet.}$$

5. Find the greatest height attained by the projectile in the last question and the time of reaching it.

Result. 400 feet above Ox, reached in 5 seconds after projection.

6. From the top, O, of a cliff 100 feet high a projectile is to be fired with a velocity of $500\ {}^f/_s$ to strike an object on the sea at a distance of 1250 feet from the vertical through O; what must be the elevation?

Result. If ϵ is the angle of elevation (*i.e.* angle between the direction of projection and the horizon) $\epsilon = 0$, or $\tan\epsilon = \frac{25}{2}$.

7. Find the angle of elevation when the range on a horizontal plane through the point of projection is equal to the height due to the velocity of projection.

Result. $\epsilon = 15°$.

8. If the range of a bullet fired on the surface of the Earth with a given velocity and elevation is R, what would be the range on the surface of the Moon with the same data, assuming that g on the Moon is $\frac{1}{6}$ of its value on the Earth?

Ans. $6R$.

9. From a given point on a horizontal plain a projectile is to be fired with a velocity of 320 $^f/_s$ to strike the top of a building 92 feet high and distant 2560 feet from the point of projection; what must be the elevation?

Ans. $\tan\epsilon = \frac{31}{16}$ or $\frac{9}{16}$.

10. In the last find the direction in which the projectile is moving when it strikes the object.

Result. When the elevation is $\tan^{-1}\frac{31}{16}$, the projectile is moving in a downward direction making $\tan^{-1}\frac{2388}{1280}$ with the horizon; and when the elevation is $\tan^{-1}\frac{9}{16}$, the direction is $\tan^{-1}\frac{157}{320}$.

11. The centre of a carriage wheel 2 feet in radius is travelling along a road with a velocity of 12 $^f/_s$; a piece of mud is thrown off the wheel at the top; will it ever meet the circumference of the wheel again?

The velocity of the highest point of the wheel is twice the velocity of the centre; hence the piece of mud is projected horizontally with a velocity of 24 $^f/_s$. Let D be the highest point of the wheel at the instant of projection; take the horizontal line through D as axis of x and the vertically downward line through D as axis of y. Then if x, y are the co-ordinates of the particle of mud after t seconds, we have

$$x = 24t;\ y = 16t^2.$$

At this time the co-ordinates of the centre of the wheel are $12t$ and 2; hence the distance between the piece of mud and the centre of the wheel is

$$\sqrt{(12t)^2 + (2 - 16t^2)^2}$$

or

$$\sqrt{256t^4 + 80t^2 + 4}.$$

Now if the piece of mud is ever again on the circumference, this distance must be 2 feet. Putting it equal to 2, we have

$$(256t^2 + 80)t^2 = 0,$$

the only solution of which is $t = 0$—*i.e.* the piece of mud is never again on the circumference.

12. If in the last the velocity of the centre of the wheel is v $^f/_s$ and the radius r feet, find the greatest velocity that will allow the particle of mud to meet the wheel again.

At any time, t, after projection the square of the distance between the centre and the particle is $v^2t^2 + (r - 16t^2)^2$, and putting this equal to r^2, we have either $t = 0$ or

$$t^2 = \frac{32r - v^2}{256}$$

$\therefore$ v must be $< 4\sqrt{2r}$ to give a real value of t.

13. A particle is projected from a point O with a velocity of 400 $^f/_s$ in a direction making $\tan^{-1}\frac{4}{3}$ with the horizon; find when, where, at what angle, and with what velocity it will strike an inclined plane passing through O, the tangent of its inclination being $\frac{5}{12}$.

Result. The time of striking is $\frac{55}{4}$ seconds. If P is the point of striking and PN the perpendicular to the horizontal plane through O, we have $ON = 240 \times \frac{55}{4} = 3300$ feet, $\therefore OP = 3575$ feet; if v_x and v_y are the horizontal and vertical components of the velocity at P, $v_x = 240$ $^f/_s$; $v_y = 120$ $^f/_s$ in the downward direction; hence the direction of motion at P makes $\tan^{-1}\frac{1}{2}$ with the horizon, and therefore the direction of motion makes $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{5}{12}$, or $\tan^{-1}\frac{22}{19}$ with the plane. The velocity of striking $= \sqrt{v_x^2 + v_y^2} = 120\sqrt{5}$ $^f/_s$.

14. A projectile is to strike at right angles an inclined plane through the point of projection; if the tangent of the inclination of this plane to the horizon is $\frac{5}{12}$, what must be the elevation of the projectile?

Ans. $\tan^{-1}\frac{97}{30}$.

15. If in the last the inclination of the inclined plane is i, what is the necessary elevation of the projectile?

Ans. $\tan^{-1}(2\tan i + \cot i)$.

16. At the same instant are projected from the same point O two particles, one with a velocity of 400 feet per second at an elevation $\tan^{-1}\frac{4}{3}$, and the other with a velocity of 340 $^f/_s$ at an elevation $\tan^{-1}\frac{8}{15}$; find their distance apart at the end of 3 seconds, and the direction of the line joining them.

Result. The distance is $180\sqrt{2}$ feet, and the line makes $45°$ with the horizon.

17. Whatever be the magnitudes and directions of the velocities of projection, the direction of the line joining two particles which are projected from the same point at the same instant remains constant during their parabolic motions.

18. A body whose mass is 26 ounces is moving horizontally with a velocity of 10 $^f/_s$ at a height of 100 feet above the ground when it is struck by a bullet whose mass is 2 ounces moving vertically upwards with a velocity of 384 $^f/_s$, the bullet remaining embedded; find the point where the conjoint mass strikes the ground.

Result. At a distance of 2080 feet from the point vertically under the position of the body when it was struck.

[Since the blows received by the two bodies are equal and opposite, the conjoint horizontal momentum after the impact is equal to that of the 26, and the conjoint vertical momentum is equal to that of the 2 ounce mass.]

19. A projectile is fired with a velocity u at an elevation ϵ from a point O (fig. 136); find when and where it will strike an inclined plane, OQ, whose inclination is i, through O.

At any time we have $y=(u\sin\epsilon-\frac{1}{2}gt)t$, and $x=u\cos\epsilon.t$. Now if the plane is struck at Q, and x, y are the co-ordinates of Q, we have $y=x\tan i$. Hence

$$u\sin\epsilon-\tfrac{1}{2}gt=u\cos\epsilon.\tan i$$

$$\therefore\ t=\frac{2u}{g}\,\frac{\sin(\epsilon-i)}{\cos i}.$$

This gives $x=\dfrac{2u^2}{g}\cdot\dfrac{\sin(\epsilon-i)}{\cos i}\cos\epsilon$; $\therefore\ OQ=\dfrac{u^2}{g}\cdot\dfrac{\sin(2\epsilon-i)-\sin i}{\cos^2 i}$.

20. What is the elevation for maximum range on an inclined plane passing through the point of projection?

The range OQ given above is a maximum, for a given value of u, when $\sin(2\epsilon-i)=1$, that is when $\epsilon=\frac{1}{2}\left(\frac{\pi}{2}-i\right)+i$, or in other words, when the direction of projection at O (fig. 136) bisects the angle yOQ. Denote this elevation by β.

21. Show that there are two directions of projection which, for a given velocity of projection, will produce a given range, OQ, on the inclined plane.

The range will be the same for the elevations ϵ and ϵ' provided that

$$2\epsilon-i+2\epsilon'-i=\pi,$$

or

$$\epsilon+\epsilon'=\frac{\pi}{2}+i$$
$$=2\beta,$$

that is, the two directions are equally inclined, on opposite sides, to the direction for maximum range.

22. A projectile is sent, with a given velocity, u, from a point O to pass through a point Q; show that the two values of the time taken in the two possible paths are given by the equation

$$g^2t^4-4(u^2-gy)t^2+4r^2=0,$$

where y is the difference of level of Q and O, and $r=OQ$.

23. If from the same point, O, and at the same instant a large number of particles are projected with the same velocity in various directions, what is the locus of the foci of their paths?

Ans. A spherical surface whose centre is O. (This is the case of a jet of water in a fountain.)

24. From a point O on a horizontal plane a projectile is sent with a velocity due to a height of 2550 feet to strike an object 288 feet above the horizontal plane through O and distant 3600 feet horizontally from O; what is the requisite elevation?

Ans. Either $\tan^{-1}\frac{8}{15}$ or $\tan^{-1}\frac{23}{10}$.

25. If in example 18 the mass of the body is W moving horizontally with 250 $^f/_s$ at a height of 216 feet, and the mass of the bullet is $\frac{1}{80}W$ striking with a vertical velocity of 1000 $^f/_s$ and remaining imbedded, in what time will the combined mass reach the ground, and at what distance from the point vertically under the point of impact?

Ans. In 4 seconds; 990 feet.

54. **Relative Velocity of two Moving Points.** — Suppose that at a given instant a point B (fig. 138) has a velocity represented by BD and that a point A has a velocity represented by AC, and let us imagine an observer moving with each point. What velocity does B *appear to* A to have? If the observer at A were reduced to rest by applying to him and to B a velocity equal and opposite to AC, appearances to A would be the same as in the actual state of motion. Hence draw Bn equal and parallel to AC, and imagine B to have the simultaneous velocities BD and Bn. Their resultant is represented by BQ, the diagonal of the parallelogram whose sides are BD and Bn; hence B's velocity *relatively* to A (considered as at rest) is represented by BQ.

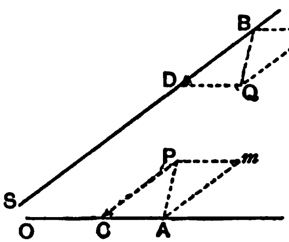


Fig. 138.

Similarly A's velocity relatively to B is obtained by reducing B to rest; and to do this we reverse the velocity BD on A and on B, so that if we draw Am equal and parallel to DB, and take the diagonal, AP, of the parallelogram whose sides are AC and Am, we have A's velocity relatively to B represented by AP; and this is exactly equal and opposite to BQ.

As an example, take the case of two trains, A and B (fig. 138), moving on two inclined lines AO, BS. If AC represents the velocity of A, and BD the velocity of B; then drawing Bn equal, parallel, and opposite to AC, and drawing the diagonal, BQ, of the parallelogram $BnQD$, the train B will *appear* to a passenger in A to move in the line BL if the velocities of the trains along AO and BS remain constant. BL is called the *relative path* of B with respect to A.

The relative velocity, at any instant, of B with respect

to A is found by the above construction. If the motions are accelerated, the relative path of B with respect to A is to be found by reversing, at each instant, not only A's *velocity* but A's *acceleration*, α, on B, and B's relative acceleration is the resultant of the two accelerations β and α, if β is B's absolute acceleration.

Consider, as a further illustration, the following problem with respect to the motions of two projectiles.

A and B (fig. 139) are two points above the ground; a particle is projected from B in a given direction, BD, with a given velocity, v; at the same instant another particle is to be projected from A so as to strike B somewhere in its path; find the direction in which the second must be projected and the requisite velocity.

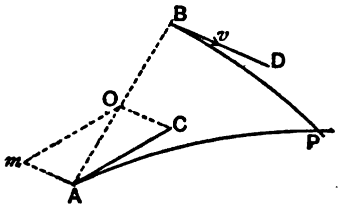


Fig. 139.

Imagine the velocity v of the particle at B to be reversed on both particles so that the particle at A receives the velocity Am equal and opposite to v; at the same time let the acceleration g, which is common to both, be removed; then the particles may just as well be considered as placed on a smooth horizontal plate at B and A, the one at B being reduced to rest; and the question comes to this: a particle is at rest at B; another at A receives a velocity, v, represented by Am, what velocity must it receive in addition to this so that it may move in the line AB? Take *any* point, O, in AB; draw Om, and AC parallel and equal to mO; then if the particle at A receives the simultaneous velocities Am and AC, its resultant velocity takes place in AO and is represented in magnitude by AO.

Hence the problem admits of an infinite number of solutions. Each particle moves in a parabola, the one in BP and the other in AP; and if A is properly projected, both particles reach P at the same instant.

Thus, in particular, if one particle is projected horizontally from the top of a building and another is at the same instant projected from the point vertically underneath on the ground with any velocity whatever which has the same horizontal

component as the velocity of the first, the particles will reach the point of intersection of their parabolic paths at the same instant.

EXAMPLES

1. A and B are two points on a horizontal plane; at the same instant two particles are projected from them in any directions; show that if the velocities of projection have the same vertical components the projectiles will reach the point common to their paths at the same instant.

2. A and B are two points in a horizontal line ABx, AB being 1280 feet; a projectile is sent from B with a velocity of 250 f/s in a direction making $\tan^{-1}\frac{3}{4}$ with Bx; and at the same instant another is sent from A in a direction making $\tan^{-1}\frac{5}{12}$ with Ax; what must be the velocity of projection of the second so that it may strike the first, and where will it strike it?

   Ans. The velocity must be 390 f/s; the particles will meet in 8 seconds after projection, at a height of 176 feet above the line ABx, and the horizontal distance travelled by the first is 1600 feet.

3. A shot of weight w is fired from a gun of weight W placed on a horizontal plane and elevated at an angle θ; prove that if the velocity with which the shot leaves the muzzle is V, the range is $\dfrac{2V^2}{g}\cdot\dfrac{\left(1+\dfrac{w}{W}\right)\tan\theta}{1+\left(1+\dfrac{w}{W}\right)^2\tan^2\theta}$.

Let fig. 140 represent the barrel and the shot. The gun recoils with a horizontal velocity u, and the relative motion of the shot with respect to the gun is obtained by reversing u on the gun and the shot, and then combining this reversed u with the velocity of the shot. Now the relative path of the shot with respect to the gun is parallel to the barrel. Hence if O is the position of the shot,

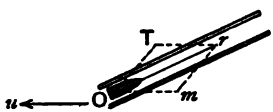


Fig. 140.

and we draw Om equal and opposite to u, and if OT represents the magnitude and direction of the velocity of the shot, the diagonal, Or, of the parallelogram determined by Om and OT lies along the barrel.

Let the horizontal and vertical components of OT be v_x and v_y; then expressing the fact that the resultant of Om and OT is along Or, we have

$$\frac{v_y}{u+v_x}=\tan\theta. \qquad (1)$$

But the horizontal impulse received by the gun from the powder pressure is equal and opposite to that received by the shot. (The vertical impulse on the gun is not the same as that received by the shot, since the ground contributes to the vertical impulse on the gun.) Hence we have

$$wv_x = Wu, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

therefore from (1)

$$v_y = \left(1 + \frac{w}{W}\right) v_x \cdot \tan\theta \quad . \quad . \quad . \quad . \quad (3)$$

$$\therefore \; V^2 = v_x^2 + v_y^2 = v_x^2 \left\{1 + \left(1 + \frac{w}{W}\right)^2 \tan^2\theta\right\}$$

$$\therefore \; v_x = \frac{V}{\left[1 + \left(1 + \frac{w}{W}\right)^2 \tan^2\theta\right]^{\frac{1}{2}}} \quad . \quad . \quad . \quad (4)$$

and (3) gives v_y. But the range is (see p. 229) $\frac{2v_x \cdot v_y}{g}$, and thus we obtain the result required. If we imagine the barrel to be transparent, OT will be the velocity of the shot as seen by an observer standing on the ground and looking at the motion of shot and gun—its direction is not along the barrel.

Directrix of Path.—Taking fig. 136, let h be the height due to the velocity, u, of projection—*i.e.* let $h = \frac{u^2}{2g}$. Measure $Oy = h$, and at y draw the horizontal line yD; this line is the directrix of the parabola described by the particle.

For, we have proved (p. 228) that the principal parameter of the curve is $\frac{2u^2 \cos^2\epsilon}{g}$, or $4h\cos^2\epsilon$; that is, the distance of the directrix above the vertex is $h\cos^2\epsilon$. But by (19), p. 230, the height VB is $h\sin^2\epsilon$; therefore the height of the directrix above the horizontal plane OA is h; that is, yD is the directrix.

The resultant velocity at each point, P, of the path is due to the depth, Pp, of the point below the directrix—that is, if v is the velocity at P,

$$v^2 = 2g \cdot pP.$$

This follows at once from the principle of work and energy; for the kinetic energy of the particle at P is $\frac{wv^2}{2g}$; at O it is

$\frac{wu^2}{2g}$; and the work done on the particle from O to P is $-wy$; therefore

$$\frac{w(v^2-u^2)}{2g} = -wy,$$

$$\therefore v^2 = u^2 - 2gy = 2g(h-y)$$
$$= 2g \, . \, pP.$$

In terms of the height h, the equation (20), p. 230, of the path referred to axes through O is

$$y = \lambda x - \frac{x^2}{4h}(1+\lambda^2),$$

where λ is tan ϵ.

EXAMINATION ON CHAPTER XIII

1. What is the path of a particle whose motion has an acceleration which is constant in magnitude and direction? If it has a constant acceleration α in one fixed direction and a constant acceleration β in another fixed direction, what is the path?

2. Considering separately the horizontal and vertical motions of a particle, write down the values, at any instant, of v_x, v_y, x, and y.

3. When can the resistance of the air be neglected in the motion of a projectile?

4. What is the direction for maximum range on a horizontal plane through the point of projection? What is it for an inclined plane passing through the point of projection?

5. How many directions of projection, with given velocity, are there for a given range on a horizontal or on an inclined plane? How are these directions related?

6. If a shot is fired horizontally from the top of a cliff, why is the time of striking the sea the same no matter what may be the velocity of projection?

7. What is meant by the relative velocity of one moving point with respect to another? How is it found?

8. Why is the relative motion of two projectiles the same as if instead of moving in a vertical plane they were projected with the same initial velocities on a smooth horizontal plane?

9. When a shot is fired from a gun which is free to recoil, why is the *elevation* at which the shot is fired different from the inclination of the barrel to the horizon?

Draw a sketch showing the state of affairs.

10. If a particle is projected in any direction with a velocity u, what is the height of the directrix of the path?

How is the velocity at any point of the path related to the directrix?

CHAPTER XIV

CALCULATION OF WORK

55. **Work Diagram.**—When a force of variable magnitude acts always in the same right line, AB (see fig. 63, p. 93), the total amount of work that it does in displacing its point of application from any one position, A, to any other position, B, is to be found by employing the principle which is used in the case of every variable magnitude (see p. 75)—viz. *however variable any magnitude may be, it can always be taken as constant throughout an infinitesimally small time or space.*

If, then, the force P varies throughout the length AB, we have merely to break up the length AB into a very great number of very small lengths—say, for example, into equal elements each $\frac{1}{100}$ inch long—and multiply the value of P at the beginning (or middle) of each of these elements, by the length of the element, and add all these products together.

The result may be very usefully represented in a diagram, such as that in fig. 55, p. 76.

Suppose, in that figure, that a variable force acts always in the line OD, and that we wish to find the amount of work which it does in displacing its point of application from M to D. Imagine the length MD broken up into small elements, MR, etc., and at the beginning of each element erect a perpendicular, MP, RQ, . . . DC, representing, on some scale, the corresponding value of the working force, P; then since the work done by the force in displacing its point of application from M to R is represented by $MP \times MR$, we easily see that the total work done by the force from M to D is graphically represented by—

the area of the diagram $MPCD$.

This area itself will be, of course, a number of square inches, but from the scale on which force has been represented, the

number of units of area can be translated into inch-pounds' weight, if we are measuring force in pounds' weight.

A curve, such as PC, whose ordinates represent the values of a working force and the elements of whose abscissæ represent displacements of the point of application (measured in the direction of the force) is called a *work diagram*, or an *indicator diagram.*

As a simple example of a variable force we shall take the tension of a stretched elastic cord.

Suppose AB (fig. 141) to be an elastic cord devoid of tension, that is at its *natural length*, and that the end A is fixed, while the end B can be seized and drawn out to various positions, C, P, D, etc. Then to each stretched length, AC, AP, AD, etc., will correspond a certain tension. The law which this tension follows is a very simple one, and it might be guessed without even making an experiment. The tension is directly proportional to the extension above the natural length. Thus, if the extension BD is 3 times the extension BC, the tension corresponding to the length AD is 3 times that corresponding to the length AC.

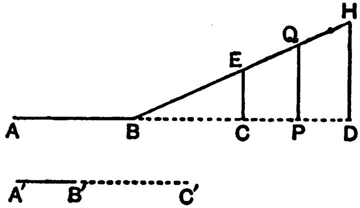


Fig. 141.

But imagine that $A'B'$ is the natural length of another piece of the same cord, and that A' is fixed while B' is drawn out, and let $A'B'$ be much shorter than AB. Then if B' is drawn out to C' so that the stretch $B'C'$ is equal to the stretch BC, it is manifest that the tension of $A'C'$ must be much greater than that of AC. We see, then, that it is not the actual amount of any extension, BC, BP, BD, etc., which measures the tension but *the ratio of this extension to the natural length*, AB; so that the tension corresponding to any length AP of the cord is proportional to $\frac{BP}{AB}$. On the same scale the tension in $A'C'$ would be proportional to $\frac{B'C'}{A'B'}$, since the cords are of the same material.

But when we have to deal with cords of *different materials*, we must introduce a coefficient which is different for different kinds of cords. If the natural (or unstretched) length of any cord AB is denoted by a, while any stretch, BP, above this length is denoted by x, the tension, T, corresponding to this stretch is expressed by the equation

$$T = \lambda\frac{x}{a}, \quad . \quad . \quad . \quad . \quad (\alpha)$$

where λ is a constant depending on the substance of the cord. This equation expresses what is known as *Hooke's Law* for an elastic cord.

The constant λ may be defined in more ways than one. Since x and a are lengths and T is force, λ must be a force.

It is—*the force necessary to double the natural length of the cord;* for if x is made equal to a, the corresponding tension equals λ. If, then, we fix one end, A, of an elastic cord which we allow to hang vertically, and from the free end, B, suspend a body of weight W which is just sufficient to double the natural length, W is equal to λ.

Or, again, we may say that λ is 100 times the force which will extend the cord by $\frac{1}{100}$ of its natural length; for if $x = \frac{a}{100}$, we have $T = \frac{\lambda}{100}$, $\therefore \lambda = 100T$.

The law (α) holds not only for cords of india-rubber but for steel (or other metallic) bars, with this obvious difference that the extensions of such bars for which the law holds are extremely small. For metal bars the first definition of λ (the force necessary to double the natural length) would not be a practical one, and we should have recourse to the second.

Now the *indicator diagram* of an elastic cord following Hooke's Law is very easily drawn. Take a numerical example. Let AB be 6 inches long, and suppose that the cord requires a force equal to 2 pounds' weight to double its natural length; take $BC = AB = 6$ inches, and at C draw an ordinate (or perpendicular), CE, to AC representing a force of 2 pounds' weight on any convenient scale; draw the right line BE and produce it indefinitely; then this line $BEQH$ represents

by its ordinates all the values of the tension of the cord corresponding to the various stretches BC, BP, BD, etc., which may be given to the cord. This is obvious, because by (a) the ordinate which represents any tension, T, is proportional to the corresponding abscissa, or stretch, x. Thus, if we wish to find the tension corresponding to a stretched length of 16 inches, we measure off the extension BP equal to 10 and then measure the ordinate PQ; thus the tension is $\frac{10}{3}$ pounds' weight.

In this numerical case what is the amount of work done against the variable tension in stretching the cord from its natural length to a length of 24 inches? Measure off BD equal to 18 inches and draw the ordinate DH to represent the corresponding tension, viz. $2 \times \frac{18}{6}$ pounds' weight—*i.e.* 6 pounds' weight; then the area of the triangle $BHD = \frac{1}{2}DH \times BD = 54$ inch-pounds' weight is the required work.

What is the work required to stretch the cord from a length of 16 to a length of 24 inches? Measure $BP = 10$ inches, and draw the ordinate PQ; then the work is represented by the area of

the trapezium $PQHD$,

i.e.

$$\tfrac{1}{2}(DH + PQ) \times PD, \text{ or } \tfrac{1}{2}(\tfrac{10}{3} + 6) \times 8,$$

or $37\frac{1}{3}$ inch-pounds' weight.

The following results are evident from the indicator diagram. The work required to stretch any elastic cord from its natural length to any final length is the same as would be done by a *constant* force equal to half the final tension working through the whole distance. The work is also the same as if a constant force equal to the final tension worked through half the distance.

As another example, take the case of the hammer and pile (fig. 74, p. 123), in which we have assumed the resistance of the ground to be a constant force. Let us now assume that the resistance is proportional to the distance, x, through which the pile has penetrated.

Suppose that after n blows of the hammer, falling each time through a height h, the ground is penetrated to a depth a, and let it be required to find the load which, placed on top of the pile, would suffice to produce the same effect by steady

pressure, the resistance being proportional to the distance penetrated.

If, as before, V is the velocity of the hammer-head on striking, the available kinetic energy after each blow is $\frac{W^2h}{W+P}$, and therefore the whole available energy employed is

$$\frac{nW^2h}{W+P}.$$

The work done by the weights of hammer and pile, assisting the penetration, is

$$(W+P)a;$$

and if we draw a diagram of the values of the resistance of the ground corresponding to values of x between 0 and a, we have such a figure as fig. 141, in which the values of x are represented by BC, BP, . . . BD, and the corresponding ground resistances by the ordinates CE, PQ, . . . DH, of a right line, BH.

The whole work done against resistance is therefore represented by the area of the triangle BDH, in which $BD=a$. If R is the final resistance, this work is, then,

$$\tfrac{1}{2}R \,.\, a.$$

Hence equating the whole positive to the whole negative work, we have

$$\frac{nW^2h}{W+P}+(W+P)a=\tfrac{1}{2}R \,.\, a$$

$$\therefore\ R=\frac{2W^2}{W+P}\cdot\frac{h}{a}+2(W+P),$$

which is double the value given by equation (6), p. 125.

The load which must be placed on top of the pile is, therefore, in this case

$$\frac{2W^2}{W+P}\cdot\frac{h}{a}+2W+P.$$

EXAMPLES

1. An elastic cord whose natural length is 5 inches can be stretched to a length of 15 inches by a force equal to 4 pounds' weight; find the amount of work necessary to extend it from a length of 5 to a length of 20 inches.

Result. 45 inch-pounds' weight.

2. Find the work necessary to stretch the cord in the last case from a length of 10 to a length of 25 inches.

Result. 75 inch-pounds' weight.

3. An elastic cord when sustaining a tension of 3 pounds' weight has a length of 14 inches, and when sustaining 9 pounds' weight has a length of 26 inches; find the work required to stretch it from a length of 20 to a length of 30 inches.

Result. 85 inch-pounds' weight.

4. An elastic cord requires an amount of work equal to 16 foot-pounds' weight to stretch it from a length of 12 to a length of 36 inches, and it requires also 16 foot-pounds' weight to stretch it from a length of 24 to a length of 40 inches; find its natural length and its elastic constant.

Result. 8 inches; 4 pounds' weight.

5. One end of the cord in Example 1 is fixed on a horizontal table (smooth), a mass of 15 ounces is attached to the other end, and this end is then drawn out until the whole length of the cord is 25 inches; if this end is now released, find the velocity of the attached mass when the cord has regained its natural length.

The tension at starting is $2 \times \frac{20}{5}$, or 8, pounds' weight, therefore the work done on the attached mass by the tension is 4×20 inch-pounds' weight, or $\frac{20}{3}$ foot-pounds' weight. Now if v f/s is the velocity with which the mass reaches the point B in fig. 141 the kinetic energy of the body is $\frac{15}{16} \cdot \frac{v^2}{64}$ foot-pounds' weight.

Equating this to the work done by the tension, we have

$$v = \tfrac{64}{3} \text{ f/s}.$$

6. One end of an elastic cord whose natural length is 8 inches and elastic constant 4 pounds' weight is fixed on a smooth horizontal table; to the other end is attached a mass of 3 ounces which is drawn out until the length of the cord is 21 inches; this end is then released; find the velocity of the attached mass when the length of the cord is 13 inches.

Result. 32 f/s.

7. An elastic cord whose natural length, AB (fig. 142), is 1 decimètre and elastic constant 1 kilogramme weight, hangs vertically from the fixed end A; to B is attached a mass of 500 grammes, and this extremity is drawn down to C, such that $AC = 5$ decimètres, and then released; find the velocity of the attached mass in any position, the position in which the velocity is greatest and the magnitude of the greatest velocity, and the greatest height attained by the mass in its upward motion.

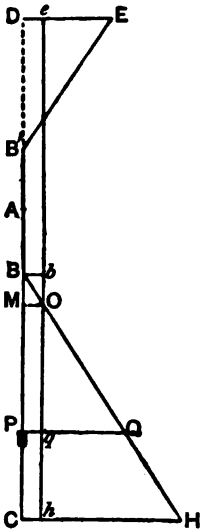


Fig. 142.

Here there are two forces acting on the body—the tension of the cord and the weight of the body. In the upward motion the former is perpetually doing positive work and the latter negative on the body. Draw the indicator diagram of each force. Measuring force in kilogrammes' weight and length in decimètres, if T is the tension of the cord in any position, P, of the mass, and x the extension BP, we have by (a), p. 244,

$$T = x \text{ kilogrammes' weight.}$$

The tension at C is therefore 4, and if we draw the ordinate CH to represent 4 kilogrammes' weight, the line BH is the indicator of tensions.

On the same scale the weight, $\frac{1}{2}$, of the body is represented by the ordinate Ch, which is $\frac{1}{8}CH$, and the indicator diagram of work done against the weight is the rectangle $BbhC$, since the weight is in all positions a constant force and it must be represented in all positions by ordinates Ch, Pq, Bb, etc., of the same length.

Now in moving from C to P, the positive work done on the body by the tension is represented by the area $CPQH$, and the negative work done on it by its weight is represented by the area $CPqh$.

If the lines hb and HB intersect in O, and from O we draw OM perpendicular to AC, the tension is always greater than the weight in the motion from C to M, and hence an excess of *positive* work is done on the body all the way from C to M, so that from C to M there is a continuous gain of kinetic energy and therefore of velocity. But after passing M, the weight exceeds the tension and an excess of *negative* work is done on the body, so that after leaving M the body continuously *loses* kinetic energy. Obviously, therefore, the velocity has its greatest value at M, *where the resultant force on the body vanishes*, *i.e.* where the body would remain at rest if it were allowed to move down very gently from B.

To find the point O, equate T to the weight, and we have

$$x=\tfrac{1}{2},$$

i.e. $BM=\tfrac{1}{2}$ decimètre.

To find the velocity at P, equate the kinetic energy to the work represented by $CPQH-CPqh$. The kinetic energy is $\dfrac{\tfrac{1}{2}\cdot v^2}{2\times 98.1}$ decimètre-kilogrammes' weight, v being taken in decimètres per second.

Now area $CPQH=\dfrac{16-x^2}{2}$, and $CPqh=\tfrac{1}{2}(4-x)$; therefore

$$\frac{v^2}{196.2}=16-x^2-(4-x)=(4-x)(3+x). \quad . \quad . \quad . \quad (1)$$

Putting $x=\tfrac{1}{2}$, we have velocity at $M=49.02$ decimètres per sec.

Putting $x=0$, we have velocity at $B=48.5$.

To find the height attained after reaching B, observe that the tension ceases, and the kinetic energy which the body had at B is now used in doing work against the weight alone, so that if y is the height attained above B, since (1) gives $\dfrac{v^2}{2g}=12$ at B, we should have $y=12$ decimètres. But this is not allowable, because when $y=2$, the cord will again become tight and assist the weight in bringing the body to rest.

Let D be the point at which the velocity ceases. Then, since the kinetic energy is zero at C and also at D, the total work done on the body by the acting forces from C to D is zero, *i.e.* the whole positive work is numerically equal to the whole negative; or, in other words,

$$\text{area } CBH=\text{area } CDeh+\text{area } B'DE.$$

If $B'D=z$, this result gives

$$z^2+z-10=0, \ \therefore \ z=2.7, \text{ nearly.}$$

Theoretically the motion would continue for ever between the extreme points C and D if no energy were lost by motion through the air and by imperfect elasticity of the cord.

8. Let the natural length of the cord AB be 3 inches, and its elastic constant 12 ounces' weight; if a mass of 3 ounces is attached to B and drawn down until the whole length of the cord is 13 inches and then let go, find the height above A that it will reach, and also its greatest velocity in the motion.

Result. It will attain a height of 11 inches above A; its greatest velocity will be 17.44 $^f/_s$, and this will take place in the same position both in the upward and in the downward motion.

9. If the natural length of the cord is a, its elastic constant λ, the weight of the attached particle w, and if the particle is drawn down until $AC=c$, show that if the motion ceases for an instant at D, we have

$$B'D=\frac{a}{\lambda}\left\{\sqrt{\left(\frac{c\lambda}{a}-w\right)^2-4\lambda w}-w\right\}.$$

10. In example 1 (p. 226), prove that if the resistance of the ground is proportional to the distance penetrated, the load which, placed on top of the pile, in addition to the hammer, is $242\frac{1}{2}$ tons.

Consider now the case in which two particles are connected by an inextensible cord, or an infinitely thin rigid bar or wire, these particles moving in any way under the action of given forces. What we wish to show is that never during the motion does the tension of the connecting cord, or the tension or pressure of the connecting bar do any total quantity of work.

Let A and B (fig. 143) be the positions of the particles at any instant, and A' and B' their positions after an indefinitely small interval of time. Suppose T to be the tension of the connecting cord or bar; then if from A' we draw $A'p$ perpendicular to AB, the work (see p. 102) which T does on the particle A in the motion from A to A' is

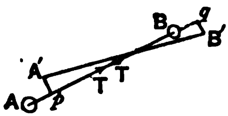


Fig. 143.

$$T\times Ap.$$

Again, if from B' we draw $B'q$ perpendicular to AB, the work which T does on the particle B in the motion from B to B' is

$$-T\times Bq.$$

Now we can see that $Ap=Bq$; because if $A'B'$ intersects AB in O, since the motions AA' and BB' are supposed to be extremely small, the angle AOA' is very small, so that $A'p$ may be considered as at right angles to both AB and $A'B'$; therefore $Op = OA'$; and similarly $Oq = OB'$; therefore $A'B' = pq$; but since the distance between the particles does not alter, $A'B'=AB$,

$$\therefore pq=AB \text{ i.e., } Ap=Bq.$$

Hence the positive work done by T on A is numerically equal to the negative work done by T on B, so that in the motion from the position AB to the position $A'B'$ the

sum of these works is zero: in other words, *the sum of the kinetic energies of the particles is not altered by* T *in this motion;* for, the gain of kinetic energy of A due to T (which is equivalent to $T \times Ap$) is exactly balanced by the loss of kinetic energy of B due to T.

The same reasoning holds for any motion, however great, of the two connected particles; for we can break up the motion into a series of small displacements—such as that from AB to $A'B'$—and in no one of these displacements does the tension or pressure of the connecting cord or bar do a total amount of work (other than zero) on the two particles.

Hence, then, *when two particles connected together by a cord or a bar whose length never alters move in any way from one position to another, their total kinetic energy is never affected by the force produced by the connection: any change of kinetic energy that takes place is due to other forces acting on them.*

Obviously the same would be true if there were *three* particles, or any number of particles, connected by cords or bars of such a nature that no particle ever alters its distance from any other one; and hence it is true for a *rigid body* of any kind, since the distance between particle and particle always remains constant—the internal tensions or pressures of the connections do not influence the kinetic energy of the body.

The kinetic energy of a body consisting of any number of particles is the sum of the kinetic energies of all its separate particles; and to this sum, if the connections are rigid, the tensions or pressures of the connections contribute nothing. Hence if such a body is moving under the action of any applied forces, its change of kinetic energy from one position to another is due entirely to these applied forces; and if its motion is the same at all times its gain of kinetic energy from one position to another is zero, so that we have the principle—

> *if a rigid body is moving so that its motion at any instant is in every way the same as at any other instant, the the total quantity of work done on it by the forces applied to it in the interval is zero.*

For example, take the case of a smooth *Screw Press* (fig. 144) working uniformly, so that its motions are the same at all

times. If P is the Effort applied to each end of the handle AB, and $AB=2a$, since the effort at A acts constantly along the tangent by the circle described by A, the work done by this force in a complete revolution of the screw is $P \times 2\pi a$; the effort at B does an equal amount of work; therefore the whole positive work done on the screw in a complete revolution is

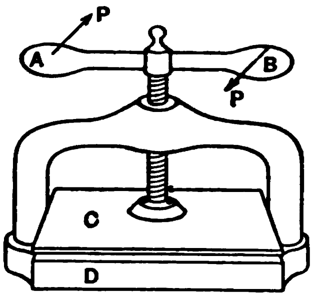


Fig. 144.

$$4\pi a.P.$$

The only other force doing work on the screw is the Resistance, which is applied to the plate C carried on the screw by the body which is being compressed between the plate C and the base, D, of the machine. Now when the handle has made a complete revolution, the end of the screw (and therefore the plate C) has travelled vertically through the pitch of the screw—that is, the distance between two successive turns of the thread measured parallel to the axis of the screw. Denote this pitch by h, and the resistance by R; then the work done by R on the screw in the motion considered is $-Rh$, so that the equation of work and energy is

$$4\pi a.P=R.h$$

$$\text{or } P=R\frac{h}{4\pi a}.$$

If i is the inclination to the horizon of the thread of the screw and r the radius of the cylinder on which the thread is cut, $h=2\pi r \tan i$; therefore

$$P=\tfrac{1}{2}R\frac{r}{a}\tan i.$$

As a final example of the calculation of work done by a variable force, consider the case of a piston driven along a cylinder by a gas expanding in volume while its temperature is kept constant.

The student will see when he comes to the study of Physics that the force exerted by the gas on the piston in these circumstances varies inversely as the volume of the

gas; that is, if F_0 is the force exerted on the piston when the volume is v_0, the force, F, exerted on the piston when the volume is v is given by the equation

$$F = F_0 \cdot \frac{v_0}{v} \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

To represent this case graphically take a line Ov (fig. 145), along which we lay off various lengths, OA, OP, . . . representing the volumes assumed by the gas; and draw a perpendicular line, Op, parallel to which we lay off, on any scale, the corresponding forces exerted on the piston by the gas. The ordinates, AM, PQ, BN, . . . which represent the values of F trace out by their extremities M, Q, N, . . . a curve called a *hyperbola* which is easily drawn from the equation (β); for if we draw AM to represent a known value, F_0, of the working force corresponding to a given volume v_0 represented by OA, and we wish to find the ordinate corresponding to any other value, v, of the volume represented by OP, we have the ordinate PQ equal to $AM \times \frac{OA}{OP}$. Thus if $OP = 4 . OA$, we have $PQ = \frac{1}{4} AM$; and so on.

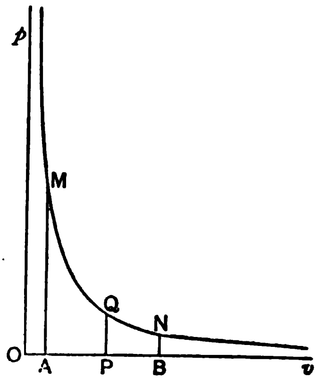


Fig. 145.

The work done by the working force F between the volumes represented by OA and OB is, then, represented by the area $AMNB$; and this can be calculated with fair accuracy by dividing the length AB into a number of equal parts—say six or so—drawing the ordinates at their extremities, connecting the upper ends of these ordinates by right lines, and taking the work as represented by the sum of the areas of a number of narrow trapeziums. The correct value of the work done by F from the value OA (or v_0) to the value OP (or v) is

$$\frac{F_0 . v_0}{S} \log_e \frac{OP}{OA}, \quad . \quad . \quad . \quad . \quad (\gamma)$$

in which S is the area of the cross-section of the cylinder and in which the logarithm is taken with respect to the Napierian base, e; and this logarithm may be replaced by one to the base 10 by division by .4343, or multiplication by 2.3025; thus the work is

$$2.3025 \times \frac{F_0 . v_0}{S} \log_{10} \frac{OP}{OA}. \quad . \quad . \quad . \quad (\delta)$$

If the student is not acquainted with the use of logarithms, he can make the calculations in the following examples by the approximate method above described.

EXAMPLES

1. A cylinder whose cross-sectional area is $\frac{1}{4}$ of a square foot is closed at one end, O, and fitted with a piston which is at a distance of 1 foot from O, the space between the piston and the end of the cylinder being filled with a gas exerting an intensity of pressure of 100 pounds' weight per square inch; if the piston is allowed to move through a length of 4 feet of the cylinder, calculate the work done on the piston by the imprisoned gas whose temperature is kept constant.

Here the initial value of the force exerted on the piston is 100×36 pounds' weight; hence when the piston is at distances of 1, 2, 3, 4 feet from O, the values of the force are 3600, 1800, 1200, and 900 pounds' weight. Making, then, an indicator diagram with ordinates 1 foot apart, and taking the whole work as represented by the areas of three trapeziums, we get the work equal to $1800 + 1800 + 1200 + 450$, or 5250, foot-pounds' weight.

The expression (δ), however, gives the work as 4990.6 foot-pounds' weight.

If instead of breaking up the whole displacement of the piston into three intervals of 1 foot each, we break it up into six, we get the series of values of F

3600, 2400, 1800, 1440, 1200, 1028.5, 900,

and the total work, calculated from 6 trapeziums is, in foot-pounds' weight,

$\frac{1}{2}(1800 + 2400 + 1800 + 1440 + 1200 + 1028.5 + 450)$,

or 5059 foot-pounds' weight, which is not very different from the true result.

2. If the cylinder in the last example has a cross-sectional area of 2 square inches, and the piston starts from a distance of $\frac{1}{2}$ inch from O, the initial intensity of pressure being 60 pounds' weight per square inch; find

the work done by the gas when the piston has moved through a length of $5\frac{1}{2}$ inches (*i.e.*, 6 inches from O).

Result. 149 inch-pounds' weight, by (8), but 152.7 by breaking up the displacement into intervals of $\frac{1}{2}$ inch.

EXAMINATION ON CHAPTER XIV

1. What is meant by a *Work Diagram*? How is the Work Diagram of a variable force constructed?
2. What is Hooke's Law for the tension of an elastic cord?
3. What is the Work Diagram of the tension of such a cord? The work done is the same as if the tension was constant—with what value?
4. What is the definition of the elastic modulus of a cord? What is the best form of the definition in the case of metal bars?
5. What is the Work Diagram of the ground resistance to a pile when the resistance is constant? When the resistance is proportional to the penetration?
6. Show that if two particles connected by a rigid wire move in any way, the tension or pressure in this wire does no work.
7. State the principle of work for any rigid body whose motion is the same at all instants.
8. What is the relation between the Effort and the Resistance in a smooth Screw Press?
9. How does the intensity of pressure of a gas vary if its volume changes but not its temperature?
10. What is the Work Diagram of a gas expanding in a cylinder under constant temperature?
11. How do you calculate approximately the total work done by the pressure of the gas on the piston from one position to another?

INDEX

W. H. WHITE AND CO. LTD.
RIVERSIDE PRESS, EDINBURGH